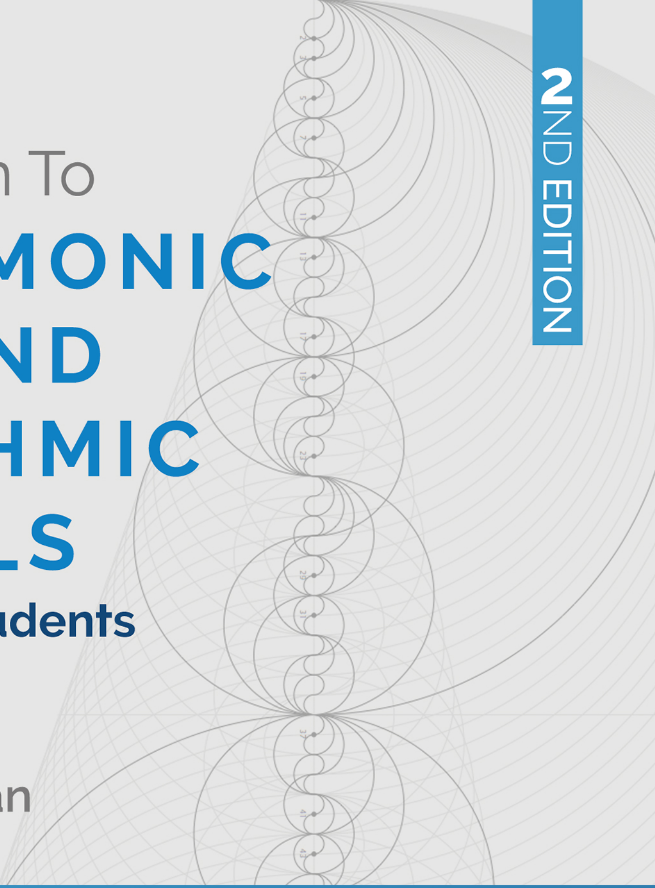
2ND EDITION



An Introduction To
THE HARMONIC SERIES AND LOGARITHMIC INTEGRALS

For High School Students
Up To Researchers

Ali Shadhar Olaikhan

$$\int_0^1 x^{n-1}\ln(1-x)\mathrm{d}x = -\frac{H_n}{n}$$

$$\left(\sum_{n=1}^{\infty} a_n x^n\right)\left(\sum_{n=1}^{\infty} b_n x^n\right) = \sum_{n=1}^{\infty} x^{n+1}\left(\sum_{k=1}^{n} a_k b_{n-k+1}\right)$$

$$\sum_{n=1}^{\infty} H_n x^n = -\frac{\ln(1-x)}{1-x}$$

$$\psi(n+1) = H_n - \gamma$$

$$\sum_{n=1}^{\infty} \frac{H_{2n}H_{2n}}{n^2} = \frac{7}{4}\zeta(4)$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\mathrm{B}(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\mathrm{d}x$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

$$\frac{\pi x}{\sin(\pi x)} = \prod_{n=1}^{\infty}\left(\frac{n^2-x^2}{n^2}\right)^{-1}$$

$$\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}\mathrm{d}t$$

$$\sum_{k=1}^{\infty} \frac{H_k^{(p)}}{k^p} = \frac{\zeta^2(p)+\zeta(2p)}{2}$$

$$\arcsin^4(x) = \frac{3}{2}\sum_{n=1}^{\infty} \frac{H_{n-1}^{(2)}(2x)^{2n}}{n^2\binom{2n}{n}}$$

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}$$

$$\mathrm{B}(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

$$\gamma = \lim_{n\to\infty}(H_n - \ln(n))$$

$$\psi(n) = \frac{\mathrm{d}}{\mathrm{d}n}\ln(\Gamma(n))$$

$$\mathrm{Li}_2^2(x) = 4\sum_{n=1}^{\infty} \frac{H_n}{n^3}x^n + 2\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2}x^n - 6\,\mathrm{Li}_4(x)$$

$$\ln|\sin x| = -\ln(2) - \sum_{n=1}^{\infty} \frac{\cos(2nx)}{n}$$

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AND LOGARITHMIC INTEGRALS

FOR HIGH SCHOOL STUDENTS
UP TO RESEARCHERS

SECOND EDITION

Ali Shadhar Olaikhan

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To my parents

Preface

The harmonic series and logarithmic integrals, which are strongly interrelated, are not typically covered in standard textbooks. Evaluating them can be challenging for new learners, as it requires specific approaches and a good understanding of special functions such as the gamma function, polygamma function, beta function, polylogarithm function, and other special functions and constants. Additionally, it requires a great deal of experience and patience, as it often involves intricate techniques and time-consuming calculations.

The purpose of this book is to introduce the harmonic series in a way that is accessible to readers with a solid foundation in calculus, from high school students to researchers.

In this edition, I have derived closed forms for the majority of the limits involved in certain results, simplified many sum expressions, improved several proofs and solutions, and added my new discoveries of integrals and harmonic series (see (3.81), (3.82), (4.38), and (4.39)).

This book is the result of over seven years of work on the harmonic series. As I taught myself about this topic, I struggled to find proofs for many of the identities required for evaluating the harmonic series. With the experience I gained over the years, I managed to prove these identities in detail using only basic definitions and well-known techniques, without relying on contour integration or the residue theorem, which require a deep understanding of complex analysis.

I would like to note that while I borrowed some proofs from various sources, mainly from the Mathematics Stack Exchange site, I modified them and added further details to suit my own approach. Additionally, most of the book is written in equations, so readers will find minimal verbiage in the text.

The book is organized into four chapters. In chapter 1, essential series transformations and special functions are presented and their relationships are explained. This chapter also provides the definition and properties of each function and derives many special values needed for the calculations in chapters 3 and 4.

Chapter 2 contains derivations of a variety of useful identities, including generating functions involving the harmonic number and series expansion of powers of $\arcsin(x)$. Other identities are derived using the binomial coefficient, the beta function, the Cauchy product, Abel's summation, and Fourier series.

Chapter 3 presents all the integral results required to calculate the harmonic series in Chapter 4, including some new results. These were derived using algebraic identities, integral manipulations, and the beta function.

Chapter 4 shows how to calculate many types of harmonic series, including non-alternating series, alternating series, series with powers of 2 in the denominator, series with powers of $2n+1$ in the denominator, series with rational argument, series with skew harmonic number, and series with central binomial coefficient. Several solutions are presented using two different methods.

At the end of the book, I have included a table of *Mathematica* commands for approximating or evaluating limits, derivatives, integrals, and series. This table enables the reader to verify any result of interest found in the book.

For more advanced and challenging problems related to the harmonic series, please visit my Mathematics Stack Exchange page at `https://math.stackexchange.com/users/432085/ali-shadhar`. I decided not to include these problems in the book for the sake of simplicity.

Finally, I would like to express my gratitude to my friend Cornel Ioan Vălean for motivating me to explore the realm of the harmonic series through his amazing problems and solutions, many of which are included in his book, *(Almost) Impossible Integrals, Sums, and Series*. I am also grateful for his valuable tips on writing this book. I would like to thank Elio Arturo and my brother Hasan Shadhar for their assistance in using LaTeX. Additionally, I extend my thanks to my friends Khalaf Ruhemi, Shivam Sharma, Wolfgang Hintze, and Hasan Hussein for their support and encouragement throughout the writing process. Lastly, I dedicate this book to my parents, who have always supported me.

Phoenix, Arizona, USA Ali Shadhar Olaikhan
March 2023

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Notations

$\mathbb{C}$	The set of complex numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{Z}$	The set of integers ($\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$)
$\mathbb{Z}_{\geq 0}$	The set of non-negative integers ($\mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}$)
$\mathbb{Z}_{\leq 0}$	The set of non-positive integers ($\mathbb{Z}_{\leq 0} = \{\dots, -2, -1, 0\}$)
$\mathbb{Z}^{+}$	The set of positive integers ($\mathbb{Z}^{+} = \{1, 2, 3, \dots\}$)
$\mathbb{Z}^{-}$	The set of negative integers ($\mathbb{Z}^{-} = \{\dots, -3, -2, -1\}$)
$\Re(z)$	The real part of a complex number z
$\Im(z)$	The imaginary part of a complex number z
$n!$	n factorial $n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n = \prod_{k=1}^{n} k, \quad n \in \mathbb{Z}^{+}$
$\binom{a}{b}$	The binomial coefficient $\binom{a}{b} = \frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)}$
$\binom{2n}{n}$	The central binomial coefficient $\binom{2n}{n} = \frac{(2n)!}{(n!)^2}$
Γ	The gamma function $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \mathrm{d}t, \quad \Re(z) > 0$
B	The beta function $\mathrm{B}(a, b) = \int_0^1 x^{a-1} (1-x)^{b-1} \mathrm{d}x, \quad \Re(a), \Re(b) > 0$
ζ	The Riemann zeta function $\zeta(z) = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \cdots = \sum_{k=1}^{\infty} \frac{1}{k^z}, \quad \Re(z) > 1$
η	The Dirichlet eta function $\eta(z) = 1 - \frac{1}{2^z} + \frac{1}{3^z} - \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^z}, \quad \Re(z) > 0$
β	The Dirichlet beta function $\beta(z) = 1 - \frac{1}{3^z} + \frac{1}{5^z} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^z}, \quad \Re(z) > 0$
Li_n	The Polylogarithm function $\mathrm{Li}_n(z) = z + \frac{z^2}{2^n} + \frac{z^3}{3^n} + \cdots = \sum_{k=1}^{\infty} \frac{z^k}{k^n}, \quad \lvert z \rvert \leq 1$
H_n	The n-th harmonic number $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}, \quad n \in \mathbb{Z}^{+}$
$H_n^{(a)}$	The n-th generalized harmonic number of order a $H_n^{(a)} = 1 + \frac{1}{2^a} + \frac{1}{3^a} + \cdots + \frac{1}{n^a} = \sum_{k=1}^{n} \frac{1}{k^a}, \quad a, n \in \mathbb{Z}^{+}$

$\overline{H}_n$	The n-th skew harmonic number $\overline{H}_n = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n-1}}{n} = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}, \quad n \in \mathbb{Z}^+$
$\overline{H}_n^{(a)}$	The n-th generalized skew harmonic number of order a $\overline{H}_n^{(a)} = 1 - \frac{1}{2^a} + \frac{1}{3^a} - \cdots + \frac{(-1)^{n-1}}{n^a} = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^a}, \quad a, n \in \mathbb{Z}^+$
ψ	The Digamma function $\psi(n) = \frac{d}{dn} \ln(\Gamma(n)) = \frac{\Gamma'(n)}{\Gamma(n)}$
$\psi^{(a)}$	The Polygamma function $\psi^{(a)}(n) = \frac{d^a}{dn^a} \psi(n) = \frac{d^{a+1}}{dn^{a+1}} \ln(\Gamma(n))$
ln	The natural logarithm ($\log_e$)
e	The base of the natural logarithm $e = \lim_{n \to \infty} (1 + 1/n)^n = 2.7182818284590\ldots$
γ	The Euler–Mascheroni constant $\gamma = \lim_{n \to \infty} (H_n - \ln(n)) = 0.5772156649015\ldots$
G	The Catalan's constant $G = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = 0.9159655941772\ldots$
E_n	The Euler number
B_n	The Bernoulli number

Chapter 1

Series Transformations and Special Functions

1.1 Index Shift Identity

The following identity holds:

$$\sum_{k=m}^{n} a_k = \sum_{k=m+c}^{n+c} a_{k-c}. \tag{1.1}$$

Proof. The index k in $\sum_{k=m}^{n} a_k$ ranges from m to n:

$$m \leq k \leq n.$$

Replace k with $j - c$,

$$m \leq j - c \leq n.$$

On solving this compound inequality, we get

$$m + c \leq j \leq n + c.$$

This indicates that if we replace the index k with $j - c$, the index j will range from $m + c$ to $n + c$:

$$\sum_{k=m}^{n} a_k = \sum_{j=m+c}^{n+c} a_{j-c}.$$

Replace j with k to finish the proof.

Example 1: Let $a_k = \frac{H_k}{k+1}$ and $m = 0$ then shift the index by -1,

$$\sum_{k=0}^{n} \frac{H_k}{k+1} = \sum_{k=1}^{n+1} \frac{H_{k-1}}{k}.$$

Example 2: Let $a_k = H_k x^{k-1}$ and $m = 3$ then shift the index by $+2$,

$$\sum_{k=3}^{n} H_k x^{k-1} = \sum_{k=1}^{n-2} H_{k+2} x^{k+1}.$$

1.2 Reversing the Order of the Sum Terms

The following identity holds:

$$\sum_{k=m}^{n} a_k = \sum_{k=m}^{n} a_{n-k+m}. \tag{1.2}$$

Proof. Applying the previous proof, we have

$$m \le k \le n.$$

Replace k with $n - j + m$,

$$m \le n + m - j \le n$$

or

$$m \le j \le n.$$

This shows that if we replace the index k with $n - j + m$, the index j will range from m to n as well:

$$\sum_{k=m}^{n} a_k = \sum_{j=m}^{n} a_{n-j+m}.$$

The proof completes on replacing j with k.

This type of transformation reverses the order of the sum terms. To see that, let $m = 1$ and $n = 4$. The LHS sum gives

$$\sum_{k=1}^{4} a_k = a_1 + a_2 + a_3 + a_4,$$

which is equivalent to the RHS sum:

$$\sum_{k=1}^{4} a_{5-k} = a_4 + a_3 + a_2 + a_1,$$

but in reversed order.

Example 1: Put $a_k = \frac{1}{k}$ and $m = 1$,

$$\sum_{k=1}^{n} \frac{1}{k} = \sum_{k=1}^{n} \frac{1}{n-k+1}. \tag{1.3}$$

Example 2: Put $a_k = k^2$ and $m = 3$,

$$\sum_{k=3}^{n} k^2 = \sum_{k=3}^{n} (n-k+3)^2.$$

1.3 Splitting a Sum Into Odd and Even Terms

Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series. Then we have

$$\sum_{n=1}^{\infty} a_n = \sum_{n=0}^{\infty} a_{2n+1} + \sum_{n=1}^{\infty} a_{2n}. \tag{1.4}$$

Proof.

$$\begin{aligned} \sum_{n=1}^{\infty} a_n &= a_1 + a_2 + a_3 + \cdots \\ &= (a_1 + a_3 + a_5 + \cdots) + (a_2 + a_4 + a_6 + \cdots) \\ &= \sum_{n=0}^{\infty} a_{2n+1} + \sum_{n=1}^{\infty} a_{2n}. \end{aligned}$$

Example 1: Put $a_n = \frac{1}{n^2}$,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2}.$$

Example 2: Put $a_n = \frac{H_n}{n^3}$,

$$\sum_{n=1}^{\infty} \frac{H_n}{n^3} = \sum_{n=0}^{\infty} \frac{H_{2n+1}}{(2n+1)^3} + \sum_{n=1}^{\infty} \frac{H_{2n}}{(2n)^3}.$$

1.4 Converting the Summand a_{2n} to a_n

Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series. Then we have

$$\sum_{n=1}^{\infty} a_{2n} = \frac{1}{2}\sum_{n=1}^{\infty} a_n + \frac{1}{2}\sum_{n=1}^{\infty} (-1)^n a_n. \tag{1.5}$$

Proof. Starting with the RHS,

$$\begin{aligned}\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} (-1)^n a^n &= a_1 + a_2 + a_3 + \cdots + (-a_1 + a_2 - a_3 + \cdots)\\ &= 2a_2 + 2a_4 + 2a_6 + \cdots\\ &= 2(a_2 + a_4 + a_6 + \cdots)\\ &= 2\sum_{n=1}^{\infty} a_{2n}.\end{aligned}$$

Applying the same approach, we also find

$$\sum_{n=0}^{\infty} a_{2n} = \frac{1}{2}\sum_{n=0}^{\infty} a_n + \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n a_n. \tag{1.6}$$

Example 1: Let $a_n = \frac{1}{(n+1)^4}$ in (1.5),

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)^4} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{(n+1)^4} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)^4}.$$

Example 2: Let $a_n = \frac{H_{n+1}}{(n+3)^3}$ in (1.6),

$$\sum_{n=0}^{\infty} \frac{H_{2n+1}}{(2n+3)^3} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{H_{n+1}}{(n+3)^3} + \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n \frac{H_{n+1}}{(n+3)^3}.$$

The following sum is related:

$$\sum_{n=1}^{\infty} a_n \cos(n\pi) = \sum_{n=1}^{\infty} (-1)^n a_n. \tag{1.7}$$

To prove it, start with LHS:

$$\begin{aligned}
\sum_{n=1}^{\infty} a_n \cos(n\pi) &= a_1 \cos(\pi) + a_2 \cos(2\pi) + a_3 \cos(3\pi) + \cdots \\
&= -a_1 + a_2 - a_3 + \cdots \\
&= \sum_{n=1}^{\infty} (-1)^n a_n.
\end{aligned}$$

1.5 Converting the Summand a_{2n+1} to a_n

Let $\sum_{n=0}^{\infty} a_n$ be an absolutely convergent series. Then we have

$$\sum_{n=0}^{\infty} a_{2n+1} = \frac{1}{2}\sum_{n=1}^{\infty} a_n - \frac{1}{2}\sum_{n=1}^{\infty} (-1)^n a_n. \tag{1.8}$$

Proof.

$$\begin{aligned}
\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} (-1)^n a^n &= a_1 + a_2 + a_3 + \cdots - (-a_1 + a_2 - a_3 + \cdots) \\
&= 2a_1 + 2a_3 + 2a_5 + \cdots \\
&= 2(a_1 + a_3 + a_5 + \cdots) \\
&= 2\sum_{n=0}^{\infty} a_{2n+1}.
\end{aligned}$$

Let’s shift the index of the LHS sum in (1.8) by -1,

$$\sum_{n=1}^{\infty} a_{2n-1} = \frac{1}{2}\sum_{n=1}^{\infty} a_n - \frac{1}{2}\sum_{n=1}^{\infty} (-1)^n a^n. \tag{1.9}$$

Example 1: Set $a_n = \frac{H_n}{n^3}$ in (1.8),

$$\sum_{n=0}^{\infty} \frac{H_{2n+1}}{(2n+1)^3} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{H_n}{n^3} - \frac{1}{2}\sum_{n=1}^{\infty} (-1)^n \frac{H_n}{n^3}.$$

Example 2: Set $a_n = \frac{1}{n^4}$ in (1.9),

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^4} - \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}.$$

1.6 Converting the Summand $(-1)^n a_{2n}$ to $i^n a_n$

Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series. Then we have

$$\sum_{n=1}^{\infty} (-1)^n a_{2n} = \Re \sum_{n=1}^{\infty} i^n a_n. \tag{1.10}$$

Proof.

$$\begin{aligned}\sum_{n=1}^{\infty} i^n a_n &= ia_1 + i^2 a_2 + i^3 a_3 + i^4 a_4 + i^5 a_5 + i^6 a_6 + \cdots \\ &= ia_1 - a_2 - ia_3 + a_4 + ia_5 - a_6 + \cdots \\ &= i(a_1 - a_3 + a_5 - \cdots) + (-a_2 + a_4 - a_6 + \cdots) \\ &= i\sum_{n=0}^{\infty} (-1)^n a_{2n+1} + \sum_{n=1}^{\infty} (-1)^n a_{2n}. \end{aligned} \tag{1.11}$$

The proof follows on comparing the real parts of both sides.

Example 1: Put $a_n = \frac{x^n}{n^3}$,

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{(2n)^3} = \Re \sum_{n=1}^{\infty} i^n \frac{x^n}{n^3}.$$

Example 2: Put $a_n = \frac{H_{n+1}}{n^2}$,

$$\sum_{n=1}^{\infty} (-1)^n \frac{H_{2n+1}}{(2n)^2} = \Re \sum_{n=1}^{\infty} i^n \frac{H_{n+1}}{n^2}.$$

A different form for the sum (1.10) is

$$\sum_{n=1}^{\infty} (-1)^n a_{2n} = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{2}\right). \tag{1.12}$$

To show that, start with the RHS:

$$\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{2}\right) = a_1 \cos\left(\frac{\pi}{2}\right) + a_2 \cos(\pi) + a_3 \cos\left(\frac{3\pi}{2}\right)$$
$$+a_4 \cos(2\pi) + a_5 \cos\left(\frac{5\pi}{2}\right) + a_6 \cos(3\pi) + \cdots$$
$$= 0 - a_2 + 0 + a_4 + 0 - a_6 + \cdots$$
$$= \sum_{n=1}^{\infty} (-1)^n a_{2n}.$$

1.7 Converting the Summand $(-1)^n a_{2n+1}$ to $i^n a_n$

Let $\sum_{n=0}^{\infty} a_n$ be an absolutely convergent series. Then we have

$$\sum_{n=0}^{\infty} (-1)^n a_{2n+1} = \Im \sum_{n=1}^{\infty} i^n a_n. \tag{1.13}$$

Proof. Compare the imaginary parts of both sides of (1.11).
Example 1: Let $a_n = \frac{1}{n^3}$,

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \Im \sum_{n=1}^{\infty} \frac{i^n}{n^3}.$$

Example 2: Let $a_n = \frac{H_n}{(n+1)^2}$,

$$\sum_{n=0}^{\infty} (-1)^n \frac{H_{2n+1}}{(2n+2)^2} = \Im \sum_{n=1}^{\infty} i^n \frac{H_n}{(n+1)^2}.$$

We also have

$$\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{2}\right) = a_1 \sin\left(\frac{\pi}{2}\right) + a_2 \sin(\pi) + a_3 \sin\left(\frac{3\pi}{2}\right)$$
$$+a_4 \sin(2\pi) + a_5 \sin\left(\frac{5\pi}{2}\right) + a_6 \sin(3\pi) + \cdots$$
$$= a_1 + 0 - a_3 + 0 + a_5 + 0 - \cdots$$
$$= \sum_{n=0}^{\infty} (-1)^n a_{2n+1}. \tag{1.14}$$

1.8 The Logarithmic Form of the Product Formula

The following identity holds:

$$\sum_{n=m}^{r} \ln(a_n) = \ln \prod_{n=m}^{r} a_n. \tag{1.15}$$

Proof.

$$\begin{aligned}\sum_{n=m}^{r} \ln(a_n) &= \ln(a_m) + \ln(a_{m+1}) + \cdots + \ln(a_r)\\ &= \ln(a_m \times a_{m+1} \times \cdots \times a_r)\\ &= \ln \prod_{n=m}^{r} a_n.\end{aligned}$$

Example 1: Let $a_n = n$,

$$\sum_{n=1}^{r} \ln(n) = \ln \prod_{n=1}^{r} n = \ln(1 \times 2 \times 3 \times \cdots \times r) = \ln(r!).$$

Example 2: Let $a_n = e^n$,

$$\ln \prod_{n=1}^{r} e^n = \sum_{n=1}^{r} \ln(e^n) = \sum_{n=1}^{r} n = \frac{r(r+1)}{2}.$$

1.9 Rearrangement Theorem for Double Series

The following identity holds:

$$\sum_{m=1}^{\infty} \sum_{n=1}^{m} a_m b_n = \sum_{n=1}^{\infty} \sum_{m=n}^{\infty} a_m b_n. \tag{1.16}$$

Proof.

$$\begin{aligned}\sum_{m=1}^{\infty} \sum_{n=1}^{m} a_m b_n &= a_1 \sum_{n=1}^{1} b_n + a_2 \sum_{n=1}^{2} b_n + a_3 \sum_{n=1}^{3} b_n + \cdots\\ &= a_1(b_1) + a_2(b_1 + b_2) + a_3(b_1 + b_2 + b_3) + \cdots\\ &= b_1(a_1 + a_2 + \cdots) + b_2(a_2 + a_3 + \cdots) + b_3(a_3 + a_4 + \cdots) + \cdots\end{aligned}$$

$$= b_1 \sum_{m=1}^{\infty} a_m + b_2 \sum_{m=2}^{\infty} a_m + b_3 \sum_{m=3}^{\infty} a_m + \cdots$$
$$= \sum_{n=1}^{\infty} b_n \sum_{m=n}^{\infty} a_m$$
$$= \sum_{n=1}^{\infty} \sum_{m=n}^{\infty} a_m b_n.$$

If we apply the same steps above, we also find

$$\sum_{m=1}^{\infty} \sum_{n=1}^{m-1} a_m b_n = \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} a_m b_n. \tag{1.17}$$

Example 1: Let $a_m = p^m$ and $b_n = p^n$,

$$\sum_{m=1}^{\infty} \sum_{n=1}^{m} p^{m+n} = \sum_{n=1}^{\infty} p^n \left(\sum_{m=n}^{\infty} p^m \right)$$
$\{$use the geometric series formula for the inner sum assuming $|p| < 1\}$
$$= \sum_{n=1}^{\infty} p^n \left(\frac{p^n}{1-p} \right)$$
$$= \frac{1}{1-p} \sum_{n=1}^{\infty} \left(p^2\right)^n$$
$\{$use the geometric series formula again$\}$
$$= \frac{1}{1-p} \left(\frac{p^2}{1-p^2} \right).$$

Example 2: Let $a_m = x^m$ and $b_n = \overline{H}_n$,

$$\sum_{m=1}^{\infty} \sum_{n=1}^{m} x^m \overline{H}_n = \sum_{n=1}^{\infty} \overline{H}_n \left(\sum_{m=n}^{\infty} x^m \right)$$
$\{$use the geometric series formula assuming $|x| < 1\}$
$$= \sum_{n=1}^{\infty} \overline{H}_n \left(\frac{x^n}{1-x} \right)$$
$$= \frac{1}{1-x} \sum_{n=1}^{\infty} \overline{H}_n x^n$$
$\{$substitute the result (2.29)$\}$

$$
\begin{aligned}
&= \frac{1}{1-x}\left(\frac{\ln(1+x)}{1-x}\right)\\
&= \frac{\ln(1+x)}{(1-x)^2}.
\end{aligned}
$$

1.10 The Logarithm of a Complex Number

The logarithm of a complex number, $z = x + iy$, is given by

$$\ln(x+iy) = \frac{1}{2}\ln(x^2+y^2) + i\arctan\left(\frac{y}{x}\right), \quad x > 0. \tag{1.18}$$

Proof. We begin with converting to polar coordinates,

$$
\begin{aligned}
x + iy &= r\cos(\theta) + ir\sin(\theta)\\
&= r(\cos(\theta) + i\sin(\theta)).
\end{aligned}
$$

On using Euler's formula:

$$e^{ix} = \cos x + i\sin x, \quad x \in \mathbb{R} \tag{1.19}$$

which can be proved by expanding $\cos x$, $\sin x$, and e^x in Taylor series, we have

$$x + iy = re^{i\theta}.$$

Take the logarithm of both sides,

$$\ln(x+iy) = \ln\left(re^{i\theta}\right) = \ln(r) + \ln\left(e^{i\theta}\right) = \ln(r) + i\theta.$$

The proof completes on substituting $r = \sqrt{x^2+y^2}$ and $\theta = \arctan(\frac{y}{x})$.
The constraint $x > 0$ in (1.18) shows that this rule is valid only when the complex number is in the first or fourth quadrant of the complex plane.
In general, for positive x and y, where $x = y \neq 0$ and $y \neq 0$, we have

$$\ln(x+iy) = \frac{1}{2}\ln(x^2+y^2) + i\arctan\left(\frac{y}{x}\right); \tag{1.20}$$

$$\ln(-x+iy) = \frac{1}{2}\ln(x^2+y^2) + i\left[\pi - \arctan\left(\frac{y}{x}\right)\right]; \tag{1.21}$$

$$\ln(-x-iy) = \frac{1}{2}\ln(x^2+y^2) - i\left[\pi - \arctan\left(\frac{y}{x}\right)\right]; \tag{1.22}$$

$$\ln(x-iy) = \frac{1}{2}\ln(x^2+y^2) - i\arctan\left(\frac{y}{x}\right). \tag{1.23}$$

For the case $y = 0$, we use

$$\ln(-x) = \ln(x) + i\pi, \quad x > 0. \tag{1.24}$$

Note that (1.22) and (1.23) follows from replacing i with $-i$ in (1.21) and (1.20) respectively. To prove (1.21), we sum it with its conjugate,

$$\ln(-x+iy) + \ln(x+iy) = \ln(-x^2 - y^2) = \ln(-1) + \ln(x^2+y^2)$$

or

$$\begin{aligned} \ln(-x+iy) &= \ln(-1) + \ln(x^2+y^2) - \ln(x+iy) \\ &\{\text{write } \ln(-1) = i\pi \text{ given in (1.25) and recall (1.20)}\} \\ &= i\pi + \ln(x^2+y^2) - \frac{1}{2}\ln(x^2+y^2) - \arctan\left(\frac{y}{x}\right) \\ &= \frac{1}{2}\ln(x^2+y^2) + i\left[\pi - \arctan\left(\frac{y}{x}\right)\right]. \end{aligned}$$

To prove (1.24), set $x = \pi$ in Euler's formula (1.19) to get

$$e^{i\pi} = -1.$$

Take the log of both sides,

$$i\pi = \ln(-1). \tag{1.25}$$

Add $\ln(x)$ to both sides,

$$\ln(x) + i\pi = \ln(x) + \ln(-1) = \ln(-x).$$

Using the rules from (1.20) to (1.23), we find:

$$\ln(1+i) = \frac{1}{2}\ln(2) + i\frac{\pi}{4}; \tag{1.26}$$

$$\ln(-1+i) = \frac{1}{2}\ln(2) + i\frac{3\pi}{4}; \tag{1.27}$$

$$\ln(-1-i) = \frac{1}{2}\ln(2) - i\frac{3\pi}{4}; \tag{1.28}$$

$$\ln(1-i) = \frac{1}{2}\ln(2) - i\frac{\pi}{4}; \tag{1.29}$$

$$\ln(i) = i\frac{\pi}{2}; \tag{1.30}$$

$$\ln(-i) = -i\frac{\pi}{2}. \tag{1.31}$$

1.11 Gamma Function

1.11.1 Definition

The gamma function is defined by

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\mathrm{d}t,\quad \Re(z)>0. \tag{1.32}$$

For a different integral form, let $t=-n\ln(x)$ and so $\mathrm{d}t=-\frac{n}{x}\mathrm{d}x$,

$$\begin{aligned}\Gamma(z)&=\int_0^\infty t^{z-1}e^{-t}\mathrm{d}t\\ &=\int_1^0 (-n\ln(x))^{z-1}e^{n\ln(x)}\left(-\frac{n}{x}\mathrm{d}x\right)\\ &=(-1)^{z-1}n^z\int_0^1 x^{n-1}\ln^{z-1}(x)\mathrm{d}x.\end{aligned}$$

Divide both sides by $n^z\Gamma(z)$,

$$\frac{1}{n^z}=\frac{(-1)^{z-1}}{\Gamma(z)}\int_0^1 x^{n-1}\ln^{z-1}(x)\mathrm{d}x,\quad \Re(z)>0. \tag{1.33}$$

1.11.2 Functional Equation

One of the key properties of the gamma function is

$$\Gamma(z+1)=z\Gamma(z),\quad z\notin\mathbb{Z}_{\le 0}. \tag{1.34}$$

Proof. Replace z with $z+1$ in (1.32),

$$\begin{aligned}\Gamma(z+1)&=\int_0^\infty t^z e^{-t}\mathrm{d}t\\ &\{\text{apply integration by parts (IBP)}\}\\ &=\underbrace{-t^z e^{-t}\Big|_0^\infty}_{0}+z\int_0^\infty t^{z-1}e^{-t}\mathrm{d}t\\ &=z\Gamma(z),\end{aligned}$$

Note that by using

$$\Gamma(1) = \int_0^\infty e^{-t}\mathrm{d}t = -e^{-t}\big|_0^\infty = -0 + 1 = 1,$$

and $\Gamma(z+1) = z\Gamma(z)$, we see that:

$$\begin{aligned}\Gamma(2) &= 1 \cdot \Gamma(1) = 1 \cdot 1 = 1!\\ \Gamma(3) &= 2 \cdot \Gamma(2) = 2 \cdot 1! = 2!\\ \Gamma(4) &= 3 \cdot \Gamma(3) = 3 \cdot 2! = 3!\\ \Gamma(5) &= 4 \cdot \Gamma(4) = 4 \cdot 3! = 4!.\end{aligned}$$

In general, we have

$$\Gamma(z) = (z-1)! \quad z \in \mathbb{Z}^+. \tag{1.35}$$

Employing this identity in (1.33) gives

$$\frac{1}{n^z} = \frac{(-1)^{z-1}}{(z-1)!}\int_0^1 x^{n-1}\ln^{z-1}(x)\mathrm{d}x, \quad z \in \mathbb{Z}^+. \tag{1.36}$$

1.11.3 Stirling's Approximation

Stirling's approximation is an approximation for factorials:

$$n! \sim \sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n}, \tag{1.37}$$

where $f(n) \sim g(n)$ means $\lim\limits_{n\to\infty}\dfrac{f(n)}{g(n)} = 1$.

The following proof may be found in [7, p. 277–278]:

Proof. Begin with the definition of the gamma function:

$$\begin{gathered}n! = \Gamma(n+1) = \int_0^\infty t^n e^{-t}\mathrm{d}t\\ \left\{\text{let } t = n + x\sqrt{2n}\right\}\\ = \sqrt{2}n^{n+\frac{1}{2}}e^{-n}\int_{-\sqrt{\frac{n}{2}}}^\infty e^{-x\sqrt{2n}}\left(1 + x\sqrt{\frac{2}{n}}\right)^n \mathrm{d}x\\ \left\{\text{write } \left(1 + x\sqrt{\frac{2}{n}}\right)^n = e^{n\ln\left(1+x\sqrt{\frac{2}{n}}\right)}\right\}\end{gathered}$$

$$= \sqrt{2}n^{n+\frac{1}{2}}e^{-n}\int_{-\sqrt{\frac{n}{2}}}^{\infty} e^{n\ln\left(1+x\sqrt{\frac{2}{n}}\right)-x\sqrt{2n}}\mathrm{d}x.$$

Divide both sides by $\sqrt{2}n^{n+\frac{1}{2}}e^{-n}$ then let $n \to \infty$, we get

$$\begin{aligned}\lim_{n\to\infty}\frac{n!}{\sqrt{2}n^{n+\frac{1}{2}}e^{-n}} &= \lim_{n\to\infty}\int_{-\sqrt{\frac{n}{2}}}^{\infty} e^{n\ln\left(1+x\sqrt{\frac{2}{n}}\right)-x\sqrt{2n}}\mathrm{d}x \\ &= \int_{-\infty}^{\infty} e^{\lim_{n\to\infty}\left[n\ln\left(1+x\sqrt{\frac{2}{n}}\right)-x\sqrt{2n}\right]}\mathrm{d}x.\end{aligned}$$

To find this limit, let $x\sqrt{\frac{2}{n}} = y$ and so $n = \frac{2x^2}{y^2}$,

$$\begin{aligned}\lim_{n\to\infty}\left[n\ln\left(1+x\sqrt{\frac{2}{n}}\right)-x\sqrt{2n}\right] &= 2x^2\lim_{y\to0}\frac{\ln(1+y)-y}{y^2} \\ &\{\text{apply L'Hôpital's rule, since we have the case } 0/0\} \\ &= 2x^2\lim_{y\to0}\frac{\frac{1}{1+y}-1}{2y} \\ &= x^2\lim_{y\to0}\frac{-1}{1+y} \\ &= -x^2.\end{aligned}$$

Plug this limit back, we get

$$\lim_{n\to\infty}\frac{n!}{\sqrt{2}n^{n+\frac{1}{2}}e^{-n}} = \int_{-\infty}^{\infty} e^{-x^2}\mathrm{d}x.$$

This integral is called the Gaussian integral, which evaluates to $\sqrt{\pi}$ by using the polar coordinates (see[44]). Thus,

$$\lim_{n\to\infty}\frac{n!}{\sqrt{2}n^{n+\frac{1}{2}}e^{-n}} = \sqrt{\pi}.$$

Divide both sides by $\sqrt{\pi}$ to finish the proof.

To evaluate the Gaussian integral in a different way, split the integral at $x = 0$,

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-x^2}\mathrm{d}x &= \underbrace{\int_{-\infty}^{0} e^{-x^2}\mathrm{d}x}_{x\to-x} + \int_{0}^{\infty} e^{-x^2}\mathrm{d}x \\ &= \int_{0}^{\infty} e^{-x^2}\mathrm{d}x + \int_{0}^{\infty} e^{-x^2}\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= 2\int_0^\infty e^{-x^2}\mathrm{d}x \\
&\overset{x=\sqrt{y}}{=} \int_0^\infty y^{-\frac{1}{2}}e^{-y}\mathrm{d}y \\
&\{\text{set } z = 1/2 \text{ in } (1.32)\} \\
&= \Gamma\left(\frac{1}{2}\right),
\end{aligned}
$$

where $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ given in (1.41).

Remark: The substitution $x \to -x$ used above means that we substitute $x = -u$. After simplifying and having everything in terms of u, we change u to x. Other common substitutions that will be frequently used are $x \to 1-x$, $x \to \frac{1}{x}$, $x \to \sqrt{x}$, $x \to x^2$, and $x \to \frac{1-x}{1+x}$.

1.11.4 Expressing the Gamma Function as a Product

The gamma function is also expressed as

$$\Gamma(z) = \frac{\Gamma(z+n+1)}{z}\prod_{k=1}^{n}\frac{1}{z+k}, \quad z \notin \mathbb{Z}_{\leq 0}. \tag{1.38}$$

Proof. By the functional equation (1.34), we have:

$$
\begin{aligned}
&\Gamma(z+1) = z\Gamma(z); \\
&\Gamma(z+2) = (z+1)\cdot\Gamma(z+1) = (z+1)\cdot z\Gamma(z); \\
&\Gamma(z+3) = (z+2)\cdot\Gamma(z+2) = (z+2)\cdot(z+1)\cdot z\Gamma(z).
\end{aligned}
$$

This can be generalized to

$$\Gamma(z+n+1) = (z+n)\cdot(z+n-1)\cdot\cdots\cdot(z+1)\cdot z\Gamma(z)$$

or

$$
\begin{aligned}
\Gamma(z) &= \frac{\Gamma(z+n+1)}{z(z+1)\cdots(z+n)} \\
&= \frac{\Gamma(z+n+1)}{z\prod_{k=1}^{n}(z+k)} \\
&= \frac{\Gamma(z+n+1)}{z}\prod_{k=1}^{n}\frac{1}{z+k}.
\end{aligned}
$$

1.11.5 Euler's Definition as an Infinite Product

Another form of the gamma function is

$$\Gamma(z) = \frac{1}{z} \prod_{k=1}^{\infty} \frac{\left(1+\frac{1}{k}\right)^z}{1+\frac{z}{k}}, \quad z \notin \mathbb{Z}_{\leq 0}. \tag{1.39}$$

Proof.

$$\begin{aligned}
\prod_{k=1}^{\infty} \frac{\left(1+\frac{1}{k}\right)^z}{1+\frac{z}{k}} &= \lim_{n\to\infty} \prod_{k=1}^{n} \frac{\left(1+\frac{1}{k}\right)^z}{\left(1+\frac{1}{n}\right)^z\left(1+\frac{z}{k}\right)} \\
&\left\{\text{use } \prod_{k=1}^{n} a_k b_k = \prod_{k=1}^{n} a_k \prod_{k=1}^{n} b_k\right\} \\
&= \lim_{n\to\infty} \prod_{k=1}^{n} \frac{\left(1+\frac{1}{k}\right)^z}{\left(1+\frac{1}{n}\right)^z} \prod_{k=1}^{n} \frac{1}{1+\frac{z}{k}} \\
&\left\{\text{use } \prod_{k=1}^{n} \frac{a_k}{a_n} = \prod_{k=1}^{n-1} a_k \text{ for the first product}\right\} \\
&= \lim_{n\to\infty} \prod_{k=1}^{n-1} \left(1+\frac{1}{k}\right)^z \prod_{k=1}^{n} \frac{k}{k+z} \\
&\left\{\text{write } \prod_{k=1}^{n-1} \left(1+\frac{1}{k}\right)^z = n^z \text{ and } \prod_{k=1}^{n} k = n!\right\} \\
&= \lim_{n\to\infty} n^z n! \prod_{k=1}^{n} \frac{1}{k+z} \\
&\{\text{this product is given in (1.38)}\} \\
&= \lim_{n\to\infty} n^z n! \frac{z\Gamma(z)}{\Gamma(n+z+1)} \\
&\{\text{use Stirling's approximation for } n! \text{ and } \Gamma(n+z+1)\} \\
&= z\Gamma(z) \lim_{n\to\infty} \frac{\sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n}n^z}{\sqrt{2\pi}(n+z)^{n+z+\frac{1}{2}}e^{-n-z}} \\
&= z\Gamma(z)e^z \lim_{n\to\infty} \frac{n^{n+z+\frac{1}{2}}}{(n+z)^{n+z+\frac{1}{2}}} \\
&= z\Gamma(z)e^z \lim_{n\to\infty} \left(\frac{n}{n+z}\right)^{z+\frac{1}{2}} \left(\frac{n}{n+z}\right)^{n}
\end{aligned}$$

$$
\begin{aligned}
&= z\Gamma(z)e^{z}\,\frac{\left(\lim\limits_{n\to\infty}\dfrac{n}{n+z}\right)^{z+\frac{1}{2}}}{\lim\limits_{n\to\infty}\left(1+\dfrac{z}{n}\right)^{n}}\\
&= z\Gamma(z)e^{z}\,\frac{(1)^{z+\frac{1}{2}}}{e^{z}}\\
&= z\Gamma(z).
\end{aligned}
$$

The identity $\prod_{k=1}^{n-1}\left(1+\frac{1}{k}\right)^{z} = z^{2}$ used in the proof can be explained as follows:

$$
\begin{aligned}
\prod_{k=1}^{n-1}\left(1+\frac{1}{k}\right)^{z} &= \prod_{k=1}^{n-1}\frac{(k+1)^{z}}{k^{z}}\\
&= \frac{\prod_{k=1}^{n-1}(k+1)^{z}}{\prod_{k=1}^{n-1}k^{z}}\\
&= \frac{2^{z}\cdot 3^{z}\cdots(n-1)^{z}\cdot n^{z}}{1^{z}\cdot 2^{z}\cdots(n-1)^{z}}\\
&= n^{z}.
\end{aligned}
$$

1.11.6 Euler's Reflection Formula

Euler's reflection formula is given by:

$$\Gamma(z)\Gamma(1-z) = \pi\csc(\pi z), \quad z \notin \mathbb{Z}. \tag{1.40}$$

Proof. Set $a = z$ and $b = 1 - z$ in the beta function (1.50):

$$\mathrm{B}(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^\infty \frac{x^{a-1}}{(1+x)^{a+b}}\mathrm{d}x,$$

we obtain

$$
\begin{aligned}
\Gamma(z)\Gamma(1-z) &= \int_0^\infty \frac{x^{z-1}}{1+x}\mathrm{d}x\\
&= \left(\int_0^1 + \int_1^\infty\right)\frac{x^{z-1}}{1+x}\mathrm{d}x\\
&= \int_0^1 \frac{x^{z-1}}{1+x}\mathrm{d}x + \underbrace{\int_1^\infty \frac{x^{z-1}}{1+x}\mathrm{d}x}_{x\to 1/x}
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^1 \frac{x^{z-1}}{1+x}\mathrm{d}x + \int_0^1 \frac{x^{-z}}{1+x}\mathrm{d}x \\
&\quad \left\{\text{expand } \frac{1}{1+x} \text{ in series in both integrals}\right\} \\
&= \int_0^1 x^{z-1}\left(\sum_{n=0}^{\infty}(-1)^n x^n\right)\mathrm{d}x + \int_0^1 x^{-z}\left(\sum_{n=0}^{\infty}(-1)^n x^n\right)\mathrm{d}x \\
&\quad \{\text{interchange integration and summation}\} \\
&= \sum_{n=0}^{\infty}(-1)^n \int_0^1 x^{n+z-1}\mathrm{d}x + \sum_{n=0}^{\infty}(-1)^n \int_0^1 x^{n-z}\mathrm{d}x \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{n+z} + \sum_{n=0}^{\infty}\frac{(-1)^n}{n-z+1} \\
&\quad \{\text{separate the first term of the first sum and shift the index of the second}\} \\
&= \frac{1}{z} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n+z} - \sum_{n=1}^{\infty}\frac{(-1)^n}{n-z} \\
&= \frac{1}{z} - 2\sum_{n=1}^{\infty}\frac{(-1)^n z}{n^2-z^2} \\
&= \frac{1}{z}\left(1 - 2\sum_{n=1}^{\infty}\frac{(-1)^n z^2}{n^2-z^2}\right) \\
&\quad \{\text{recall the identity (2.98)}\} \\
&= \frac{1}{z}\left(\pi z \csc(\pi z)\right) \\
&= \pi \csc(\pi z).
\end{aligned}
$$

Setting $z = \frac{1}{2}$ in (1.40) yields

$$\Gamma^2\left(\frac{1}{2}\right) = \pi \csc\left(\frac{\pi}{2}\right) = \pi$$

or

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{1.41}$$

Remark: The interchange of integration and summation used in the proof above is justified by Lebesgue's dominated convergence theorem (see [13]):

$$\int_X \sum_{n=1}^{\infty} f_n(x)\mathrm{d}x = \sum_{n=1}^{\infty}\int_X f_n(x)\mathrm{d}x, \tag{1.42}$$

where X is the integration interval.

This theorem will be frequently used throughout the book where it's applicable. So we are not going to re-explain why the interchange is justified if used again.

1.11.7 Legendre Duplication Formula

The Legendre duplication formula is given by

$$\Gamma\left(\frac{1}{2}+n\right)=\frac{2\sqrt{\pi}\,\Gamma(2n)}{4^n\Gamma(n)}. \tag{1.43}$$

The following proof may be found in [45]:

Proof. Letting $a=b=n$ in the beta function in (1.46):

$$\mathrm{B}(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}=\int_0^1 x^{a-1}(1-x)^{b-1}\mathrm{d}x$$

yields

$$\begin{aligned}
\frac{\Gamma^2(n)}{\Gamma(2n)} &= \int_0^1 x^{n-1}(1-x)^{n-1}\mathrm{d}x\\
&\overset{x=\frac{1+u}{2}}{=} 2^{1-2n}\int_{-1}^1 (1-u^2)^{n-1}\mathrm{d}u\\
&= 2^{1-2n}\left(\int_{-1}^0+\int_0^1\right)(1-u^2)^{n-1}\mathrm{d}u\\
&= 2^{1-2n}\underbrace{\int_{-1}^0 (1-u^2)^{n-1}\mathrm{d}u}_{u\to -u}+2^{1-2n}\int_0^1 (1-u^2)^{n-1}\mathrm{d}u\\
&= 2^{1-2n}\int_0^1 (1-u^2)^{n-1}\mathrm{d}u+2^{1-2n}\int_0^1 (1-u^2)^{n-1}\mathrm{d}u\\
&= 2^{1-2n}\int_0^1 2(1-u^2)^{n-1}\mathrm{d}u \qquad (1.44)\\
&\overset{u=\sqrt{x}}{=} 2^{1-2n}\int_0^1 x^{-\frac{1}{2}}(1-x)^{n-1}\mathrm{d}x
\end{aligned}$$

{use the definition of the beta function (1.46)}

$$= 2^{1-2n}\,\mathrm{B}\left(\frac{1}{2},n\right)$$

{recall the property of the beta function (1.47)}

$$= \frac{2\,\Gamma(\frac{1}{2})\Gamma(n)}{4^n\Gamma(\frac{1}{2}+n)}.$$

The proof completes on substituting $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ given in (1.41).
Further, substitute $u = \cos(x)$ in (1.44) to have

$$\int_0^{\frac{\pi}{2}} \sin^{2n-1}(x)\mathrm{d}x = \frac{4^n\Gamma^2(n)}{4\Gamma(2n)}.$$

Let's simplify the RHS. By the definition of the binomial coefficient:

$$\binom{a}{b} = \frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)},$$

we have

$$\begin{aligned}\binom{2n}{n} &= \frac{\Gamma(2n+1)}{\Gamma^2(n+1)}\\ &\{\text{use } \Gamma(z+1) = z\Gamma(z) \text{ given in (1.34)}\}\\ &= \frac{2\Gamma(2n)}{n\Gamma^2(n)}\end{aligned}$$

or

$$\frac{\Gamma^2(n)}{\Gamma(2n)} = \frac{2}{\binom{2n}{n}n}.$$

Thus,

$$\int_0^{\frac{\pi}{2}} \sin^{2n-1}(x)\mathrm{d}x = \frac{4^n}{\binom{2n}{n}}\frac{1}{2n}. \tag{1.45}$$

More results of the gamma function may be found in [43].

1.12 Beta Function

1.12.1 Definition

The beta function is defined by

$$\mathrm{B}(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\mathrm{d}x, \quad \Re(a),\, \Re(b) > 0. \tag{1.46}$$

One of the key identities of the beta function is the Beta–Gamma identity:

$$\mathrm{B}(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \tag{1.47}$$

The following proof may be found in [47]:

Proof. Using the definition of the gamma function in (1.32), we have

$$\begin{aligned}\Gamma(a)\Gamma(b)&=\left(\int_0^\infty e^{-x}x^a\mathrm{d}x\right)\left(\int_0^\infty e^{-y}y^b\mathrm{d}y\right)\\&\{\text{let } x=zt \text{ and } y=z(1-t)\}\\&=\int_0^\infty\int_0^1 e^{-z}(zt)^{a-1}(z(1-t))^{b-1}z\,\mathrm{d}t\,\mathrm{d}z\\&=\left(\int_0^\infty e^{-z}z^{a+b-1}\mathrm{d}z\right)\left(\int_0^1 t^{a-1}(1-t)^{b-1}\mathrm{d}t\right)\\&=\Gamma(a+b)\,\mathrm{B}(a,b).\end{aligned}$$

The proof follows on dividing both sides by $\Gamma(a+b)$.

Another identity is the symmetry:

$$\mathrm{B}(a,b)=\mathrm{B}(b,a). \tag{1.48}$$

To prove it, let $1-x=y$ in (1.46),

$$\begin{aligned}\mathrm{B}(a,b)&=\int_0^1 x^{a-1}(1-x)^{b-1}\mathrm{d}x\\&=\int_0^1 y^{b-1}(1-y)^{a-1}\mathrm{d}y\\&=\mathrm{B}(b,a).\end{aligned}$$

1.12.2 Trigonometric Integral Representation

A different integral form of the beta function is

$$\mathrm{B}(a,b)=2\int_0^{\frac{\pi}{2}}\cos^{2a-1}(x)\sin^{2b-1}(x)\mathrm{d}x,\quad \Re(a),\,\Re(b)>0. \tag{1.49}$$

Proof. Let $x=\cos^2(\theta)$ in (1.46) and so $\mathrm{d}x=-2\cos(\theta)\sin(\theta)\mathrm{d}\theta$,

$$\mathrm{B}(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}\mathrm{d}x$$

$$= \int_{\frac{\pi}{2}}^{0} \left(\cos^2(\theta)\right)^{a-1} \left(1-\cos^2(\theta)\right)^{b-1} (-2\cos(\theta)\sin(\theta)\mathrm{d}\theta)$$

$$= \int_{\frac{\pi}{2}}^{0} \left(\cos^2(\theta)\right)^{a-1} \left(\sin^2(\theta)\right)^{b-1} (-2\cos(\theta)\sin(\theta)\mathrm{d}\theta)$$

$$= 2\int_{0}^{\frac{\pi}{2}} \cos^{2a-1}(\theta)\sin^{2b-1}(\theta)\mathrm{d}\theta.$$

1.12.3 Improper Integral Representation

Another form of the beta function is

$$\mathrm{B}(a,b) = \int_0^\infty \frac{x^{a-1}}{(1+x)^{a+b}}\mathrm{d}x, \quad \Re(a),\, \Re(b) > 0. \tag{1.50}$$

Proof. Let $x = \frac{y}{1+y}$ in (1.46) and so $\mathrm{d}x = \frac{\mathrm{d}y}{(1+y)^2}$,

$$\mathrm{B}(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\mathrm{d}x$$

$$= \int_0^\infty \left(\frac{y}{1+y}\right)^{a-1} \left(1-\frac{y}{1+y}\right)^{b-1} \frac{\mathrm{d}y}{(1+y)^2}$$

$$= \int_0^\infty \left(\frac{y}{1+y}\right)^{a-1} \left(\frac{1}{1+y}\right)^{b-1} \frac{\mathrm{d}y}{(1+y)^2}$$

$$= \int_0^\infty \frac{y^{a-1}}{(1+y)^{a+b}}\mathrm{d}y.$$

1.12.4 Powerful Integral Representation

We also have

$$\mathrm{B}(a,b) = \int_0^1 \frac{x^{a-1}+x^{b-1}}{(1+x)^{a+b}}\mathrm{d}x, \quad \Re(a),\, \Re(b) > 0. \tag{1.51}$$

The following proof may be found in [32, p. 72]:
Proof. Let $x \to \frac{1}{x}$,

$$\int_0^1 \frac{x^{a-1}+x^{b-1}}{(1+x)^{a+b}}\mathrm{d}x = \int_1^\infty \frac{x^{a-1}+x^{b-1}}{(1+x)^{a+b}}\mathrm{d}x$$

$$\left\{\text{add } \int_0^1 \frac{x^{a-1}+x^{b-1}}{(1+x)^{a+b}}\mathrm{d}x \text{ to both sides then divide by } 2\right\}$$

$$= \frac{1}{2}\int_0^1 \frac{x^{a-1}+x^{b-1}}{(1+x)^{a+b}}\mathrm{d}x + \frac{1}{2}\int_1^\infty \frac{x^{a-1}+x^{b-1}}{(1+x)^{a+b}}\mathrm{d}x$$

$$= \frac{1}{2}\left(\int_0^1 + \int_1^\infty\right) \frac{x^{a-1}+x^{b-1}}{(1+x)^{a+b}}\mathrm{d}x$$

$$= \frac{1}{2}\int_0^\infty \frac{x^{a-1}+x^{b-1}}{(1+x)^{a+b}}\mathrm{d}x$$

$$= \frac{1}{2}\int_0^\infty \frac{x^{a-1}}{(1+x)^{a+b}}\mathrm{d}x + \frac{1}{2}\int_0^\infty \frac{x^{b-1}}{(1+x)^{a+b}}\mathrm{d}x$$

{use the definition of the beta function (1.50)}

$$= \frac{1}{2}\,\mathrm{B}(a,b) + \frac{1}{2}\,\mathrm{B}(b,a)$$

{write $\mathrm{B}(b,a) = \mathrm{B}(a,b)$ given in (1.48)}

$$= \frac{1}{2}\,\mathrm{B}(a,b) + \frac{1}{2}\,\mathrm{B}(a,b)$$

$$= \mathrm{B}(a,b).$$

1.13 Dirichlet Eta Function

1.13.1 Definition

The Dirichlet eta function is defined by

$$\eta(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z}, \quad \Re(z) > 0. \tag{1.52}$$

1.13.2 Integral Representation

The integral representation of the eta function is

$$\eta(z) = \frac{(-1)^{z-1}}{\Gamma(z)} \int_0^1 \frac{\ln^{z-1}(x)}{1+x}\mathrm{d}x, \quad \Re(z) > 0. \tag{1.53}$$

Proof. By using (1.33):

$$\frac{1}{n^z} = \frac{(-1)^{z-1}}{\Gamma(z)} \int_0^1 x^{n-1} \ln^{z-1}(x) \mathrm{d}x,$$

we have

$$\begin{aligned}
\eta(z) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z} = \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{(-1)^{z-1}}{\Gamma(z)} \int_0^1 x^{n-1} \ln^{z-1}(x) \mathrm{d}x \right) \\
&\qquad \{\text{interchange integration and summation}\} \\
&= \frac{(-1)^{z-1}}{\Gamma(z)} \int_0^1 \ln^{z-1}(x) \left(\sum_{n=1}^{\infty} (-x)^{n-1} \right) \mathrm{d}x \\
&\qquad \{\text{use the geometric series formula}\} \\
&= \frac{(-1)^{z-1}}{\Gamma(z)} \int_0^1 \ln^{z-1}(x) \left(\frac{1}{1+x} \right) \mathrm{d}x.
\end{aligned}$$

1.13.3 Evaluation of $\eta(0)$

The following result is very essential for later calculations:

$$\eta(0) = \frac{1}{2}. \tag{1.54}$$

The following proof may be found in [9]:

Proof (i). Let's start with the integral form of the Dirichlet eta function,

$$\begin{aligned}
\eta(z) &= \frac{(-1)^{z-1}}{\Gamma(z)} \int_0^1 \frac{\ln^{z-1}(x)}{1+x} \mathrm{d}x \\
&\overset{x=e^{-t}}{=} \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{t^{z-1}}{1+e^t} \mathrm{d}t \\
&\overset{\text{IBP}}{=} \frac{1}{\Gamma(z)} \left(\underbrace{\left. \frac{t^z}{z(1+e^t)} \right|_0^{\infty}}_{0} + \frac{1}{z} \int_0^{\infty} \frac{t^z e^t}{(1+e^t)^2} \mathrm{d}t \right) \\
&\qquad \{\text{write } z\Gamma(z) = \Gamma(z+1)\} \\
&= \frac{1}{\Gamma(z+1)} \int_0^{\infty} \frac{t^z e^t}{(1+e^t)^2} \mathrm{d}t.
\end{aligned}$$

Take the limit on both sides letting $z \to 0^+$,

$$\eta(0) = \frac{1}{\Gamma(1)} \int_0^\infty \frac{e^t}{(1+e^t)^2} \mathrm{d}t = -\frac{1}{1+e^t}\bigg|_0^\infty = \frac{1}{2}.$$

Proof (ii).

$$\begin{aligned}
\eta(0) &= \lim_{s \to 0^+} \eta(s) \\
&= \lim_{s \to 0^+} \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^s} \\
&= \lim_{s \to 0^+} \left(\lim_{x \to 1^-} \sum_{n=1}^\infty \frac{(-x)^{n-1}}{n^s} \right) \\
&\{\text{change the order of the limits}\} \\
&= \lim_{x \to 1^-} \left(\lim_{s \to 0^+} \sum_{n=1}^\infty \frac{(-x)^{n-1}}{n^s} \right) \\
&= \lim_{x \to 1^-} \left(\sum_{n=1}^\infty (-x)^{n-1} \right) \\
&= \lim_{x \to 1^-} \left(\frac{1}{1+x} \right) = \frac{1}{2}.
\end{aligned}$$

A justification for this proof may be found in [20].

1.14 Riemann Zeta Function

1.14.1 Definition

The Riemann zeta function is defined by

$$\zeta(z) = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \cdots = \sum_{n=1}^\infty \frac{1}{n^z}, \quad \Re(z) > 1. \tag{1.55}$$

The zeta function $\zeta(z)$ can be related to the Dirichlet eta function $\eta(z)$ by setting $a_n = \frac{1}{n^z}$ in (1.5):

$$\sum_{n=1}^\infty a_n = 2\sum_{n=1}^\infty a_{2n} - \sum_{n=1}^\infty (-1)^n a_n,$$

which gives us

$$\sum_{n=1}^{\infty}\frac{1}{n^z}=2\sum_{n=1}^{\infty}\frac{1}{(2n)^z}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^z}.$$

Rearrange the terms and write $\sum_{n=1}^{\infty}\frac{1}{n^z}=\zeta(z)$,

$$\zeta(z)=\frac{1}{1-2^{1-z}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^z}.$$

Since the series on the RHS is defined for $\Re(z)>0$, the domain of $\zeta(z)$ is extended to $\Re(z)>0$, meaning that

$$\zeta(z)=\frac{\eta(z)}{1-2^{1-z}}\quad \Re(z)>0,\ z\neq 1. \tag{1.56}$$

The input value $z=1$ is excluded, because it makes the factor $\frac{1}{1-2^{1-z}}$ undefined.

Examples:

$$\begin{aligned}\zeta(2)&=2\eta(2);\\ \zeta(3)&=\frac{4}{3}\eta(3);\\ \zeta(4)&=\frac{8}{7}\eta(4);\\ \zeta(5)&=\frac{16}{15}\eta(5).\end{aligned}$$

To find the value of $\zeta(0)$, let $z\to 0^+$ in (1.56),

$$\zeta(0)=-\eta(0)=-\frac{1}{2}, \tag{1.57}$$

where we used $\eta(0)=\frac{1}{2}$ given in (1.54).

1.14.2 Integral Representation

The integral form of the Riemann zeta function is

$$\zeta(z)=\frac{(-1)^{z-1}}{\Gamma(z)}\int_0^1\frac{\ln^{z-1}(x)}{1-x}\mathrm{d}x,\quad \Re(z)>1. \tag{1.58}$$

Proof. Using (1.36):

$$\frac{1}{n^z} = \frac{(-1)^{z-1}}{\Gamma(z)} \int_0^1 x^{n-1} \ln^{z-1}(x) \mathrm{d}x,$$

we have

$$\begin{aligned}
\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} &= \sum_{n=1}^{\infty} \left(\frac{(-1)^{z-1}}{\Gamma(z)} \int_0^1 x^{n-1} \ln^{z-1}(x) \mathrm{d}x \right) \\
&\{\text{interchange integration and summation}\} \\
&= \frac{(-1)^{z-1}}{\Gamma(z)} \int_0^1 \ln^{z-1}(x) \left(\sum_{n=1}^{\infty} x^{n-1} \right) \mathrm{d}x \\
&\{\text{use the geometric series formula}\} \\
&= \frac{(-1)^{z-1}}{\Gamma(z)} \int_0^1 \ln^{z-1}(x) \left(\frac{1}{1-x} \right) \mathrm{d}x.
\end{aligned}$$

1.14.3 Evaluation of $\zeta(2)$

Leonhard Euler was the first one to prove

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{1.59}$$

The following proof is different from Euler's and it may be found in [3]:

Proof. Squaring both sides of

$$\frac{\pi}{4} = \int_0^1 \frac{\mathrm{d}x}{1+x^2},$$

we have

$$\begin{aligned}
\frac{\pi^2}{16} &= \left(\int_0^1 \frac{\mathrm{d}y}{1+y^2} \right) \left(\int_0^1 \frac{\mathrm{d}x}{1+x^2} \right) \\
&= \int_0^1 \int_0^1 \frac{\mathrm{d}y \, \mathrm{d}x}{(1+x^2)(1+y^2)} \\
&\overset{y=t/x}{=} \int_0^1 \int_0^x \frac{\mathrm{d}t \, \mathrm{d}x}{x(1+x^2)(1+t^2/x^2)} \\
&\{\text{change the order of integration}\}
\end{aligned}$$

$$= \int_0^1 \int_t^1 \frac{\mathrm{d}x\,\mathrm{d}t}{x(1+x^2)(1+t^2/x^2)}$$

$$\overset{x=\sqrt{u}}{=} \frac{1}{2}\int_0^1 \left[\int_{t^2}^1 \frac{\mathrm{d}u}{(1+u)(u+t^2)}\right]\mathrm{d}t$$

$$\left\{\text{write } \frac{1}{(1+u)(u+t^2)} = \frac{1}{1-t^2}\left(\frac{1}{u+t^2}-\frac{1}{1+u}\right)\right\}$$

$$= \frac{1}{2}\int_0^1 \frac{1}{1-t^2}\left[\int_{t^2}^1 \frac{\mathrm{d}u}{u+t^2} - \int_{t^2}^1 \frac{\mathrm{d}u}{1+u}\right]\mathrm{d}t$$

$$= \frac{1}{2}\int_0^1 \frac{1}{1-t^2}\left[\ln(u+t^2)-\ln(1+u)\right]_{u=t^2}^{u=1}\mathrm{d}t$$

$$= \frac{1}{2}\int_0^1 \frac{1}{1-t^2}\left[\ln\left(\frac{1+t^2}{2}\right)-\ln\left(\frac{2t^2}{1+t^2}\right)\right]\mathrm{d}t$$

$$= \frac{1}{2}\int_0^1 \frac{1}{1-t^2}\ln\left(\frac{(1+t^2)^2}{4t^2}\right)\mathrm{d}t$$

$$= \int_0^1 \frac{\ln\left(\frac{1+t^2}{2t}\right)}{1-t^2}\mathrm{d}t$$

$$\overset{t=\frac{1-x}{1+x}}{=} \frac{1}{2}\int_0^1 \frac{\ln\left(\frac{1+x^2}{1-x^2}\right)}{x}\mathrm{d}x$$

$$\overset{x=\sqrt{y}}{=} \frac{1}{4}\int_0^1 \frac{\ln\left(\frac{1+y}{1-y}\right)}{y}\mathrm{d}y$$

$$= \frac{1}{4}\int_0^1 \frac{\ln\left(\frac{1-y^2}{(1-y)^2}\right)}{y}\mathrm{d}y$$

$$= \frac{1}{4}\underbrace{\int_0^1 \frac{\ln(1-y^2)}{y}\mathrm{d}y}_{y\to\sqrt{y}} - \frac{1}{2}\int_0^1 \frac{\ln(1-y)}{y}\mathrm{d}y$$

$$= \frac{1}{8}\int_0^1 \frac{\ln(1-y)}{y}\mathrm{d}y - \frac{1}{2}\int_0^1 \frac{\ln(1-y)}{y}\mathrm{d}y$$

$$= -\frac{3}{8}\int_0^1 \frac{\ln(1-y)}{y}\mathrm{d}y$$

$$\{\text{expand } \ln(1-y) \text{ in series}\}$$

$$= -\frac{3}{8}\int_0^1 \frac{1}{y}\left(-\sum_{n=1}^{\infty}\frac{y^n}{n}\right)\mathrm{d}y$$

$$\{\text{interchange integration and summation}\}$$

$$
\begin{aligned}
&= \frac{3}{8}\sum_{n=1}^{\infty}\frac{1}{n}\int_0^1 y^{n-1}\mathrm{d}y \\
&= \frac{3}{8}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{n}\right) \\
&= \frac{3}{8}\zeta(2).
\end{aligned}
$$

So we have $\frac{\pi^2}{16} = \frac{3}{8}\zeta(2)$ or $\zeta(2) = \frac{\pi^2}{6}$ and the proof is complete.

1.14.4 Evaluation of $\zeta(2n)$

Let $n \in \mathbb{Z}^+$. Then the following identity holds:

$$\zeta(2n) = \frac{(-1)^{n+1}B_{2n}(2\pi)^{2n}}{2(2n)!}. \tag{1.60}$$

Proof. Separating the first term using $\zeta(0) = -\frac{1}{2}$, we have

$$
\begin{gathered}
\sum_{n=0}^{\infty}\zeta(2n)x^{2n} = -\frac{1}{2} + \sum_{n=1}^{\infty}\zeta(2n)x^{2n} \\
\{\text{expand } \zeta(2n) \text{ in series}\} \\
= -\frac{1}{2} + \sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty}\frac{1}{k^{2n}}\right)x^{2n} \\
\{\text{change the order of summations}\} \\
= -\frac{1}{2} + \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\left(x^2/k^2\right)^n\right) \\
\{\text{use the geometric series formula}\} \\
= -\frac{1}{2} + \sum_{k=1}^{\infty}\frac{x^2/k^2}{1-x^2/k^2} \\
= -\frac{1}{2}\left(1 - 2\sum_{k=1}^{\infty}\frac{x^2}{k^2-x^2}\right) \\
\{\text{use the result (2.96)}\} \\
= -\frac{1}{2}\pi x\cot(\pi x) \\
\{\text{replace } x \text{ with } \pi x \text{ in (1.170)}\}
\end{gathered}
$$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n B_{2n} (2\pi)^{2n}}{2(2n)!} x^{2n}.$$

So we have

$$\sum_{n=0}^{\infty} \zeta(2n) x^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{(2n)!} x^{2n}.$$

The proof finishes on comparing the coefficients of x^{2n}.

Examples:

$$\begin{aligned} \zeta(4) &= \frac{\pi^4}{90}; \\ \zeta(6) &= \frac{\pi^6}{945}; \\ \zeta(8) &= \frac{\pi^8}{9450}; \\ \zeta(10) &= \frac{\pi^{10}}{93555}. \end{aligned}$$

1.15 Dirichlet Beta Function

1.15.1 Definition

The Dirichlet beta function is defined by

$$\beta(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^z} = 1 - \frac{1}{3^z} + \frac{1}{5^z} - \cdots, \quad \Re(z) > 0. \tag{1.61}$$

1.15.2 Integral Representation

The integral form of the Dirichlet beta function is

$$\beta(z) = \frac{(-1)^{z-1}}{\Gamma(z)} \int_0^1 \frac{\ln^{z-1}(x)}{1+x^2} \mathrm{d}x, \quad \Re(z) > 0. \tag{1.62}$$

Proof. By writing

$$\frac{1}{(2n+1)^z} = \frac{(-1)^{z-1}}{\Gamma(z)} \int_0^1 x^{2n} \ln^{z-1}(x) \mathrm{d}x,$$

which follows from (1.33), we have

$$\begin{aligned}\beta(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^z} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{(-1)^{z-1}}{\Gamma(z)} \int_0^1 x^{2n} \ln^{z-1}(x) \mathrm{d}x \right) \\ &= \frac{(-1)^{z-1}}{\Gamma(z)} \int_0^1 \ln^{z-1}(x) \left(\sum_{n=0}^{\infty} (-x^2)^n \right) \mathrm{d}x \\ &= \frac{(-1)^{z-1}}{\Gamma(z)} \int_0^1 \frac{\ln^{z-1}(x)}{1+x^2} \mathrm{d}x.\end{aligned}$$

1.15.3 Dirichlet Beta–Polygamma Relation

Let $a \in \mathbb{Z}^+$. Then we have

$$\beta(a) = \frac{(-1)^a 2^{1-2a}}{(a-1)!} \psi^{(a-1)}\left(\frac{1}{4}\right) - \left(1 - \frac{1}{2^a}\right) \zeta(a). \tag{1.63}$$

Proof.

$$\begin{aligned}\beta(a) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^a} \\ &\left\{ \text{use} \sum_{n=0}^{\infty} (-1)^n a_n = 2 \sum_{n=0}^{\infty} a_{2n} - \sum_{n=0}^{\infty} a_n \text{ given in (1.6)} \right\} \\ &= 2 \sum_{n=0}^{\infty} \frac{1}{(4n+1)^a} - \sum_{n=0}^{\infty} \frac{1}{(2n+1)^a}.\end{aligned}$$

The first sum:

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{(4n+1)^a} &= \frac{1}{4^a} \sum_{n=0}^{\infty} \frac{1}{(n+1/4)^a} \\ &\{\text{make use of the definition (1.152)}\} \\ &= \frac{(-1)^a 2^{-2a}}{(a-1)!} \psi^{(a-1)}\left(\frac{1}{4}\right).\end{aligned} \tag{1.64}$$

The second sum: Set $a_n = \frac{1}{(n+1)^a}$ in (1.6),

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^a} &= \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{(n+1)^a} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^a}\\ &\{\text{shift the index } n \text{ by } -1 \text{ in both series on the RHS}\}\\ &= \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^a} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^a}\\ &= \frac{1}{2}\zeta(a) + \frac{1}{2}\eta(a).\end{aligned}$$

Substitute $\eta(a) = (1-2^{1-a})\zeta(a)$ given in (1.56), we have

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^a} = (1-2^{-a})\zeta(a). \tag{1.65}$$

The proof completes on collecting the two sums.

Examples:

$$\begin{aligned}\beta(2) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} &= -\int_0^1 \frac{\ln(x)}{1+x^2}\mathrm{d}x = \frac{\psi^{(1)}\left(\frac{1}{4}\right)}{48} - \frac{3}{4}\zeta(2);\\ \beta(4) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^4} &= -\frac{1}{6}\int_0^1 \frac{\ln^3(x)}{1+x^2}\mathrm{d}x = \frac{\psi^{(3)}\left(\frac{1}{4}\right)}{768} - \frac{15}{16}\zeta(4);\\ \beta(6) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^6} &= -\frac{1}{120}\int_0^1 \frac{\ln^5(x)}{1+x^2}\mathrm{d}x = \frac{\psi^{(5)}\left(\frac{1}{4}\right)}{245760} - \frac{63}{64}\zeta(6);\\ \beta(8) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^8} &= -\frac{1}{5040}\int_0^1 \frac{\ln^7(x)}{1+x^2}\mathrm{d}x = \frac{\psi^{(7)}\left(\frac{1}{4}\right)}{165150720} - \frac{255}{256}\zeta(8).\end{aligned}$$

1.15.4 Evaluation of $\beta(2a+1)$

Let $a \in \mathbb{Z}_{\geq 0}$. Then we have

$$\beta(2a+1) = \frac{|E_{2a}|}{2(2a)!}\left(\frac{\pi}{2}\right)^{2a+1}. \tag{1.66}$$

Proof. Replace z with $2a+1$ in (1.62) then write $\Gamma(2a+1) = (2a)!$,

$$(2a)!\beta(2a+1) = \int_0^1 \frac{\ln^{2a}(x)}{1+x^2}\mathrm{d}x$$

$$
\begin{aligned}
&= \left(\int_0^\infty - \int_1^\infty\right) \frac{\ln^{2a}(x)}{1+x^2}\mathrm{d}x \\
&= \int_0^\infty \frac{\ln^{2a}(x)}{1+x^2}\mathrm{d}x - \underbrace{\int_1^\infty \frac{\ln^{2a}(x)}{1+x^2}\mathrm{d}x}_{x\to 1/x} \\
&= \int_0^\infty \frac{\ln^{2a}(x)}{1+x^2}\mathrm{d}x - \int_0^1 \frac{\ln^{2a}(x)}{1+x^2}\mathrm{d}x \\
&\qquad \{\text{rearrange the terms}\} \\
&= \frac{1}{2}\int_0^\infty \frac{\ln^{2a}(x)}{1+x^2}\mathrm{d}x.
\end{aligned}
$$

This integral is given in (3.21).

Examples:

$$
\begin{aligned}
\beta(1) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \int_0^1 \frac{\mathrm{d}x}{1+x^2} = \frac{\pi}{4}; \\
\beta(3) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{1}{2}\int_0^1 \frac{\ln^2(x)}{1+x^2}\mathrm{d}x = \frac{\pi^3}{32}; \\
\beta(5) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^5} = \frac{1}{24}\int_0^1 \frac{\ln^4(x)}{1+x^2}\mathrm{d}x = \frac{5\pi^5}{1536}; \\
\beta(7) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^7} = \frac{1}{720}\int_0^1 \frac{\ln^6(x)}{1+x^2}\mathrm{d}x = \frac{16\pi^7}{184320}.
\end{aligned}
$$

1.16 Polylogarithm Function

1.16.1 Definition

The polylogarithm function of order a is defined by

$$
\mathrm{Li}_a(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^a} = z + \frac{z^2}{2^a} + \frac{z^3}{3^a} + \cdots, \quad |z| \le 1, \tag{1.67}
$$

where $|z| = \sqrt{x^2+y^2}$ is the modulus of the complex number, $z = x + iy$.

Let's discuss the case $a = 1$:

$$\mathrm{Li}_1(z) = \sum_{n=1}^{\infty} \frac{z^n}{n} = -\ln(1-z). \tag{1.68}$$

This function diverges when $|z| = 1$ and converges to $-\ln(2)$ when $z = -1$, and so the case $a = 1$ is valid when $z = -1$, but invalid when $|z| = 1$. Actually the range of a can be extended to the whole complex plane. To keep it simple, we will consider only the case $a \in \mathbb{Z}^+$, since that is all we need for later calculations.

Note that

$$\mathrm{Li}_a(1) = \sum_{n=1}^{\infty} \frac{(1)^n}{n^a} = \sum_{n=1}^{\infty} \frac{1}{n^a} = \zeta(a) \tag{1.69}$$

and

$$\mathrm{Li}_a(-1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^a} = -\eta(a) = (2^{1-a} - 1)\zeta(a). \tag{1.70}$$

Examples:

$$\begin{aligned} \mathrm{Li}_2(-1) &= -\frac{1}{2}\zeta(2); \\ \mathrm{Li}_3(-1) &= -\frac{3}{4}\zeta(3); \\ \mathrm{Li}_4(-1) &= -\frac{7}{8}\zeta(4); \\ \mathrm{Li}_5(-1) &= -\frac{15}{16}\zeta(5). \end{aligned}$$

It's also valid to set $z = i$ in (1.67), since $|i| = 1$ assuming $a = 2, 3, 4, \cdots$,

$$\begin{aligned} \mathrm{Li}_a(i) &= \sum_{n=1}^{\infty} \frac{i^n}{n^a} \\ &\left\{\text{use } \sum_{n=1}^{\infty} f(n) = \sum_{n=0}^{\infty} f(2n+1) + \sum_{n=1}^{\infty} f(2n) \text{ given in (1.4)}\right\} \\ &= \sum_{n=0}^{\infty} \frac{i^{2n+1}}{(2n+1)^a} + \sum_{n=1}^{\infty} \frac{i^{2n}}{(2n)^a} \\ &= i\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^a} + 2^{-a} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^a} \\ &\{\text{use the definitions (1.61) and (1.52)}\} \\ &= i\beta(a) - 2^{-a}\eta(a) \end{aligned}$$

$$\begin{aligned}&\{\text{write } \eta(a) = (1-2^{1-a})\zeta(a) \text{ given in (1.56)}\}\\ &= i\beta(a) + (2^{1-2a} - 2^{-a})\zeta(a).\end{aligned}$$

Thus,

$$\operatorname{Li}_a(i) = (2^{1-2a} - 2^{-a})\zeta(a) + i\beta(a). \tag{1.71}$$

Examples:

$$\operatorname{Li}_2(i) = -\frac{1}{8}\zeta(2) + i\beta(2) = -\frac{1}{8}\zeta(2) + iG; \tag{1.72}$$

$$\operatorname{Li}_3(i) = -\frac{3}{32}\zeta(3) + i\beta(3) = -\frac{3}{32}\zeta(3) + i\frac{\pi^3}{32}; \tag{1.73}$$

$$\operatorname{Li}_4(i) = -\frac{7}{128}\zeta(4) + i\beta(4). \tag{1.74}$$

1.16.2 Integral Representation

Let $z \in \mathbb{C} \setminus (1, \infty)$. Then we have

$$\operatorname{Li}_a(z) = \frac{(-1)^{a-1}}{(a-1)!}\int_0^1 \frac{z\ln^{a-1}(t)}{1-zt}\mathrm{d}t. \tag{1.75}$$

The following proof may be found in [32, p. 70–72]:

Proof. Force integration by parts,

$$\begin{aligned}&\int_0^1 \frac{z\ln^{a-1}(t)}{1-zt}\mathrm{d}t\\ &= -\ln(1-zt)\ln^{a-1}(t)\Big|_{t=0}^{t=1} + (a-1)\int_0^1 \frac{\ln(1-zt)\ln^{a-2}(t)}{t}\mathrm{d}t\\ &= 0 + (a-1)\int_0^1 \frac{\ln(1-zt)\ln^{a-2}(t)}{t}\mathrm{d}t\\ &\overset{\text{IBP}}{=} (a-1)\left(-\operatorname{Li}_2(zt)\ln^{a-2}(t)\Big|_{t=0}^{t=1} + (a-2)\int_0^1 \frac{\operatorname{Li}_2(zt)\ln^{a-3}(t)}{t}\mathrm{d}t\right)\\ &= (a-1)(a-2)\int_0^1 \frac{\operatorname{Li}_2(zt)\ln^{a-3}(t)}{t}\mathrm{d}t\\ &\overset{\text{IBP}}{=} -(a-1)(a-2)(a-3)\int_0^1 \frac{\operatorname{Li}_3(zt)\ln^{a-4}(t)}{t}\mathrm{d}t\\ &\{\text{apply integration by parts for another } a-4 \text{ times}\}\end{aligned}$$

$$\begin{aligned}&= (-1)^{a-1}(a-1)(a-2)(a-3)\cdots 1 \int_0^1 \frac{\mathrm{Li}_{a-1}(zt)}{t}\mathrm{d}t\\ &= (-1)^{a-1}(a-1)!\,\mathrm{Li}_a(zt)\Big|_{t=0}^{t=1}\\ &= (-1)^{a-1}(a-1)!\,\mathrm{Li}_a(z).\end{aligned}$$

Let's replace z with $\frac{z}{z-1}$ in (1.75),

$$\mathrm{Li}_a\left(\frac{z}{z-1}\right) = \frac{(-1)^a}{(a-1)!}\int_0^1 \frac{z\ln^{a-1}(t)}{1-z+zt}\mathrm{d}t, \quad z\in\mathbb{C}\setminus[\,1\,,\ \infty)\,. \tag{1.76}$$

A different integral form is

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{z^n}{n^a} = \sum_{n=1}^{\infty}\frac{1}{n^{a-1}}\frac{z^n}{n} &= \sum_{n=1}^{\infty}\frac{1}{n^{a-1}}\left(\int_0^z t^{n-1}\mathrm{d}t\right)\\ &= \int_0^z \frac{1}{t}\left(\sum_{n=1}^{\infty}\frac{t^n}{n^{a-1}}\right)\mathrm{d}t\\ &= \int_0^z \frac{1}{t}\left(\mathrm{Li}_{a-1}(t)\right)\mathrm{d}t.\end{aligned}$$

Thus,

$$\mathrm{Li}_a(z) = \int_0^z \frac{\mathrm{Li}_{a-1}(t)}{t}\mathrm{d}t, \quad z\in\mathbb{C}. \tag{1.77}$$

This proof is valid if $|z|\le 1$, since we used the definition $\sum_{n=1}^{\infty}\frac{z^n}{n^a}$, but the integral form extends the range of z to the whole complex plane.

By using the integral representation of $\mathrm{Li}_a(z)$ and $\mathrm{Li}_a(-z)$, we find

$$\begin{aligned}\mathrm{Li}_a(z)+\mathrm{Li}_a(-z) &= \frac{(-1)^{a-1}}{(a-1)!}\int_0^1 \ln^{a-1}(t)\left(\frac{z}{1-zt}-\frac{z}{1+zt}\right)\mathrm{d}t\\ &= \frac{(-1)^{a-1}}{(a-1)!}\int_0^1 \ln^{a-1}(t)\left(\frac{2tz^2}{1-z^2t^2}\right)\mathrm{d}t\\ &\overset{t=\sqrt{y}}{=} 2^{1-a}\frac{(-1)^{a-1}}{(a-1)!}\int_0^1 \frac{z^2\ln^{a-1}(y)}{1-z^2y}\mathrm{d}y\\ &\quad \{\text{replace } z \text{ with } z^2 \text{ in (1.75) to get this integral}\}\\ &= 2^{1-a}\,\mathrm{Li}_a(z^2).\end{aligned}$$

Then, we have

$$\mathrm{Li}_a(z)+\mathrm{Li}_a(-z) = 2^{1-a}\,\mathrm{Li}_a(z^2), \quad z\in\mathbb{C}. \tag{1.78}$$

A different approach, assuming $|z| \leq 1$, is

$$\begin{aligned}
\operatorname{Li}_a(z) + \operatorname{Li}_a(-z) &= \sum_{n=1}^{\infty} \frac{z^n}{n^a} + \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n^a} \\
&\left\{\text{use} \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} (-1)^n a_n = 2\sum_{n=1}^{\infty} a_{2n} \text{ given in (1.5)}\right\} \\
&= 2\sum_{n=1}^{\infty} \frac{z^{2n}}{(2n)^a} \\
&= 2^{1-a} \sum_{n=1}^{\infty} \frac{(z^2)^n}{n^a} \\
&= 2^{1-a} \operatorname{Li}_a(z^2).
\end{aligned}$$

Let's differentiate the polylogarithm function with respect to z,

$$\begin{aligned}
\frac{\partial}{\partial z} \operatorname{Li}_a(z) &= \frac{\partial}{\partial z} \sum_{n=1}^{\infty} \frac{z^n}{n^a} \\
&\{\text{differentiate term by term}\} \\
&= \sum_{n=1}^{\infty} \frac{n z^{n-1}}{n^a} \\
&= \frac{1}{z} \sum_{n=1}^{\infty} \frac{z^n}{n^{a-1}} \\
&= \frac{\operatorname{Li}_{a-1}(z)}{z}. \qquad (1.79)
\end{aligned}$$

1.16.3 Dilogarithm Reflection Formula

Let $z \in \mathbb{C}$. Then the following identity holds:

$$\operatorname{Li}_2(z) + \operatorname{Li}_2(1-z) = \zeta(2) - \ln(z)\ln(1-z). \qquad (1.80)$$

Proof. Differentiate $\operatorname{Li}_2(1-z)$ then integrate,

$$\begin{aligned}
\operatorname{Li}_2(1-z) &= \int \mathrm{d}(\operatorname{Li}_2(1-z)) \\
&= \int \frac{-\operatorname{Li}_1(1-z)}{1-z} \mathrm{d}z \\
&\{\text{use } \operatorname{Li}_1(x) = -\ln(1-x) \text{ given in (1.68)}\}
\end{aligned}$$

$$
\begin{aligned}
&= \int \frac{\ln(z)}{1-z} \mathrm{d}z \\
&\overset{\text{IBP}}{=} -\ln(1-z)\ln(z) + \int \frac{\ln(1-z)}{z} \mathrm{d}z \\
&= -\ln(1-z)\ln(z) - \operatorname{Li}_2(z) + c.
\end{aligned}
$$

To find c, let $z = 1$,

$$\operatorname{Li}_2(0) = 0 - \operatorname{Li}_2(1) + c$$

or $c = \operatorname{Li}_2(1) = \zeta(2)$ and the proof is complete.
Setting $z = \frac{1}{2}$ in (1.80) gives

$$\operatorname{Li}_2\left(\frac{1}{2}\right) = \frac{1}{2}\zeta(2) - \frac{1}{2}\ln^2(2). \tag{1.81}$$

Additionally, set $z = i$ in (1.80),

$$\operatorname{Li}_2(i) + \operatorname{Li}_2(1-i) = \zeta(2) - \ln(i)\ln(1-i).$$

Substitute the values (1.30), (1.29), and (1.72) to have

$$\operatorname{Li}_2(1-i) = \frac{3}{8}\zeta(2) - i\left(\frac{\pi}{4}\ln(2) + G\right). \tag{1.82}$$

Replacing i with $-i$ in (1.82) yields

$$\operatorname{Li}_2(1+i) = \frac{3}{8}\zeta(2) + i\left(\frac{\pi}{4}\ln(2) + G\right). \tag{1.83}$$

1.16.4 Landen's Dilogarithm Identity

Let $z \in \mathbb{C} \setminus \{0\}$. Then the following identity holds:

$$\operatorname{Li}_2(1-z) + \operatorname{Li}_2\left(\frac{z-1}{z}\right) = -\frac{1}{2}\ln^2(z). \tag{1.84}$$

Proof. Differentiate $\operatorname{Li}_2\left(\frac{z-1}{z}\right)$ then integrate,

$$
\begin{aligned}
\operatorname{Li}_2\left(\frac{z-1}{z}\right) &= \int \mathrm{d}\left(\operatorname{Li}_2\left(\frac{z-1}{z}\right)\right) \\
&= \int \frac{\operatorname{Li}_1\left(\frac{z-1}{z}\right)}{z(1-z)} \mathrm{d}z \\
&= -\int \frac{\ln(z)}{z(1-z)} \mathrm{d}z
\end{aligned}
$$

$$= -\int \frac{\ln(z)}{z}\mathrm{d}z - \int \frac{\ln(z)}{1-z}\mathrm{d}z$$
$$= -\frac{1}{2}\ln^2(z) - \mathrm{Li}_2(1-z) + c.$$

The proof finishes on extracting $c = 0$ by putting $z = 1$.
Let's set $z = 1 - i$ in (1.84),

$$\mathrm{Li}_2(i) + \mathrm{Li}_2\left(\frac{1-i}{2}\right) = -\frac{1}{2}\ln^2(1-i).$$

Substitute the values (1.72), (1.30), and (1.29),

$$\mathrm{Li}_2\left(\frac{1-i}{2}\right) = \frac{5}{16}\zeta(2) - \frac{1}{8}\ln^2(2) - i\left(G - \frac{\pi}{8}\ln(2)\right). \tag{1.85}$$

Replace i with $-i$,

$$\mathrm{Li}_2\left(\frac{1+i}{2}\right) = \frac{5}{16}\zeta(2) - \frac{1}{8}\ln^2(2) + i\left(G - \frac{\pi}{8}\ln(2)\right). \tag{1.86}$$

Other way to find this result is by setting $z = -1 + i$ in (1.87).

1.16.5 Dilogarithm Inversion Formula

Let $z \in \mathbb{C} \setminus \{0\}$. Then the following formula holds:

$$\mathrm{Li}_2(-z) + \mathrm{Li}_2\left(-\frac{1}{z}\right) = -\frac{\ln^2(z)}{2} - 2\eta(2). \tag{1.87}$$

Proof.

$$\mathrm{Li}_2(-z) + \mathrm{Li}_2\left(-\frac{1}{z}\right) = \int \mathrm{d}\left(\mathrm{Li}_2(-z) + \mathrm{Li}_2\left(-\frac{1}{z}\right)\right)$$
$$= \int \left(-\frac{\ln(1+z)}{z} + \frac{\ln\left(\frac{1+z}{z}\right)}{z}\right)\mathrm{d}z$$
$$= -\int \frac{\ln(z)}{z}\mathrm{d}z$$
$$= -\frac{1}{2}\ln^2(z) + c.$$

Set $z = 1$, we get $c = 2\,\mathrm{Li}_2(-1) = -2\eta(2)$ and the proof is finalized.

1.16.6 Relation Involving Four Dilogarithm Functions

Let $z \in \mathbb{C} \setminus \{-1\}$. Then the following identity holds:

$$\begin{aligned}\operatorname{Li}_2\left(\frac{2z}{1+z}\right) - \operatorname{Li}_2\left(\frac{z}{1+z}\right) + \operatorname{Li}_2\left(\frac{1+z}{2}\right)\\ = \operatorname{Li}_2(z) + \operatorname{Li}_2\left(\frac{1}{2}\right) + \ln(2)\ln(1+z).\end{aligned} \tag{1.88}$$

Proof. Let

$$f(z) = \operatorname{Li}_2\left(\frac{2z}{1+z}\right) - \operatorname{Li}_2\left(\frac{z}{1+z}\right) + \operatorname{Li}_2\left(\frac{1+z}{2}\right),$$

we have

$$\begin{aligned}f(z) = \int \mathrm{d}\, f(z) = \int \left(\frac{\ln(2)}{1+z} - \frac{\ln(1-z)}{z}\right)\mathrm{d}z\\ = \ln(2)\ln(1+z) + \operatorname{Li}_2(z) + c.\end{aligned}$$

Setting $z = 0$ gives $c = f(1) = \operatorname{Li}_2(\frac{1}{2})$ and the proof is finalized.

1.16.7 Another Relation Involving Dilogarithm Functions

Let $z \in \mathbb{C} \setminus \{-1\}$. Then the following identity holds:

$$\begin{aligned}\operatorname{Li}_2\left(\frac{2z}{1+z}\right) - \operatorname{Li}_2\left(\frac{z}{1+z}\right) - \operatorname{Li}_2\left(\frac{1-z}{2}\right) - \operatorname{Li}_2(z) + \operatorname{Li}_2\left(\frac{1}{2}\right)\\ = \ln(1-z)\ln\left(\frac{1+z}{2}\right).\end{aligned} \tag{1.89}$$

Proof. Let

$$f(z) = \operatorname{Li}_2\left(\frac{2z}{1+z}\right) - \operatorname{Li}_2\left(\frac{z}{1+z}\right) - \operatorname{Li}_2\left(\frac{1-z}{2}\right) - \operatorname{Li}_2(z) + \operatorname{Li}_2\left(\frac{1}{2}\right),$$

we have

$$\begin{aligned}f(z) = \int \mathrm{d}\, f(z) = \int \left(\frac{\ln(1-z)}{1+z} - \frac{\ln(1+z)}{1-z} + \frac{\ln(2)}{1-z}\right)\mathrm{d}z\\ = \int \left(\frac{\ln(1-z)}{1+z} - \frac{\ln(1+z)}{1-z}\right)\mathrm{d}z + \int \frac{\ln(2)}{1-z}\mathrm{d}z\end{aligned}$$

$$\begin{aligned}&= \int \mathrm{d}(\ln(1-z)\ln(1+z)) - \ln(2)\ln(1-z)\\ &= \ln(1-z)\ln(1+z) - \ln(2)\ln(1-z)\\ &= \ln(1-z)\ln\left(\frac{1+z}{2}\right) + c.\end{aligned}$$

The proof completes on extracting $c = 0$ by setting $z = 0$.

1.16.8 Landen's Trilogarithm Identity

Let $z \in \mathbb{C} \setminus \{0\}$. Then the following identity holds:

$$\begin{aligned}\mathrm{Li}_3(z) + \mathrm{Li}_3(1-z) + \mathrm{Li}_3\left(\frac{z-1}{z}\right) &= \zeta(3) + \frac{1}{6}\ln^3(z) + \zeta(2)\ln(z)\\ &-\frac{1}{2}\ln^2(z)\ln(1-z). \qquad (1.90)\end{aligned}$$

Proof.

$$\begin{aligned}\mathrm{Li}_3\left(\frac{z-1}{z}\right) &= \int \mathrm{d}\left(\mathrm{Li}_3\left(\frac{z-1}{z}\right)\right)\\ &= \int -\frac{\mathrm{Li}_2\left(\frac{z-1}{z}\right)}{z(1-z)}\mathrm{d}z\end{aligned}$$

{make use of Landen's dilogarithm identity (1.84)}

$$\begin{aligned}&= \int \left(\frac{\mathrm{Li}_2(1-z)}{z(1-z)} + \frac{\ln^2(z)}{2z(1-z)}\right)\mathrm{d}z\\ &= \int \left(\frac{\mathrm{Li}_2(1-z)}{z} + \frac{\mathrm{Li}_2(1-z)}{1-z} + \frac{1}{2}\frac{\ln^2(z)}{z} + \frac{1}{2}\frac{\ln^2(z)}{1-z}\right)\mathrm{d}z\\ &= \underbrace{\int \frac{\mathrm{Li}_2(1-z)}{x}\mathrm{d}z}_{\text{IBP}} - \mathrm{Li}_3(1-z) + \frac{1}{6}\ln^3(z) + \frac{1}{2}\int \frac{\ln^2(z)}{1-z}\mathrm{d}z\\ &= \ln(z)\,\mathrm{Li}_2(1-z) - \mathrm{Li}_3(1-z) + \frac{1}{6}\ln^3(z) - \frac{1}{2}\int \frac{\ln^2(z)}{1-z}\mathrm{d}z.\end{aligned}$$

For the remaining integral, force integration by parts twice,

$$\begin{aligned}\int \frac{\ln^2(z)}{1-z}\mathrm{d}z &= -\ln(1-z)\ln^2(z) + 2\int \frac{\ln(1-z)\ln(z)}{z}\mathrm{d}z\\ &= -\ln(1-z)\ln^2(z) - 2\,\mathrm{Li}_2(z)\ln(z) + 2\int \frac{\mathrm{Li}_2(z)}{z}\mathrm{d}z\end{aligned}$$

$$= -\ln(1-z)\ln^2(z) - 2\operatorname{Li}_2(z)\ln(z) + 2\operatorname{Li}_3(z). \tag{1.91}$$

Plug this integral back in,

$$\begin{aligned}
\operatorname{Li}_3\left(\frac{z-1}{z}\right) &= \ln(z)[\operatorname{Li}_2(1-z)+\operatorname{Li}_2(z)] + \frac{1}{2}\ln(1-z)\ln^2(z) + \frac{1}{6}\ln^3(z) \\
&\quad - \operatorname{Li}_3(1-z) - \operatorname{Li}_3(z) \\
&\quad \{\text{make use of (1.80) for the first term}\} \\
&= \ln(z)[\zeta(2) - \ln(x)\ln(1-x)] + \frac{1}{2}\ln(1-z)\ln^2(z) + \frac{1}{6}\ln^3(z) \\
&\quad - \operatorname{Li}_3(1-z) - \operatorname{Li}_3(z) \\
&= \zeta(2)\ln(z) - \frac{1}{2}\ln(1-z)\ln^2(z) + \frac{1}{6}\ln^3(z) - \operatorname{Li}_3(1-z) - \operatorname{Li}_3(z) + c
\end{aligned}$$

and the proof follows on extracting $c = \zeta(3)$ by setting $z = 1$.

Setting $z = \frac{1}{2}$ in (1.90) yields

$$\operatorname{Li}_3\left(\frac{1}{2}\right) + \operatorname{Li}_3\left(\frac{1}{2}\right) + \operatorname{Li}_3(-1) = \zeta(3) - \frac{1}{6}\ln^3(2) - \zeta(2)\ln(2) + \frac{1}{2}\ln^3(2).$$

Substitute $\operatorname{Li}_3(-1) = -\frac{3}{4}\zeta(3)$ given in (1.70),

$$\operatorname{Li}_3\left(\frac{1}{2}\right) = \frac{7}{8}\zeta(3) - \frac{1}{2}\ln(2)\zeta(2) + \frac{1}{6}\ln^3(2). \tag{1.92}$$

For another result, set $z = i$ in (1.90) then consider the real parts of both sides,

$$\begin{aligned}
&\Re\left\{\operatorname{Li}_3(i) + \operatorname{Li}_3(1-i) + \operatorname{Li}_3\left(\frac{i-1}{i}\right)\right\} \\
&= \zeta(3) + \Re\left\{\frac{1}{6}\ln^3(i) + \zeta(2)\ln(i) - \frac{1}{2}\ln^2(i)\ln(1-i)\right\}.
\end{aligned}$$

Since $\operatorname{Li}_3\left(\frac{i-1}{i}\right) = \operatorname{Li}_3(1+i)$ and $\Re\operatorname{Li}_3(1-i) = \Re\operatorname{Li}_3(1+i)$, we have

$$\begin{aligned}
&\Re\{\operatorname{Li}_3(i) + 2\operatorname{Li}_3(1+i)\} \\
&= \zeta(3) + \Re\left\{\frac{1}{6}\ln^3(i) + \zeta(2)\ln(i) - \frac{1}{2}\ln^2(i)\ln(1-i)\right\}.
\end{aligned}$$

Collect the results from (1.73), (1.30), and (1.29),

$$\Re\operatorname{Li}_3(1+i) = \frac{3}{16}\ln(2)\zeta(2) + \frac{35}{64}\zeta(3) = \Re\operatorname{Li}_3(1-i). \tag{1.93}$$

1.16.9 First Polylogarithm Inversion Formula

Let $z \in \mathbb{C} \setminus \{0\}$. Then the following two formulas hold:

$$\operatorname{Li}_{2a}(-z) + \operatorname{Li}_{2a}\left(-\frac{1}{z}\right) = -2\sum_{k=0}^{a} \frac{\eta(2a-2k)}{(2k)!} \ln^{2k}(z); \tag{1.94}$$

$$\operatorname{Li}_{2a+1}(-z) - \operatorname{Li}_{2a+1}\left(-\frac{1}{z}\right) = -2\sum_{k=0}^{a} \frac{\eta(2a-2k)}{(2k+1)!} \ln^{2k+1}(z). \tag{1.95}$$

Proof. Divide both sides of (1.87) by z then integrate from $z = 1$ to z,

$$\operatorname{Li}_3(-z) - \operatorname{Li}_3\left(-\frac{1}{z}\right) = -\frac{\ln^3(z)}{3!} - 2\eta(2)\ln(z). \tag{1.96}$$

Repeat the same process,

$$\operatorname{Li}_4(-z) + \operatorname{Li}_4\left(-\frac{1}{z}\right) = -\frac{\ln^4(z)}{4!} - 2\eta(2)\frac{\ln^2(z)}{2} - 2\eta(4), \tag{1.97}$$

again,

$$\operatorname{Li}_5(-z) - \operatorname{Li}_5\left(-\frac{1}{z}\right) = -\frac{\ln^5(z)}{5!} - 2\eta(2)\frac{\ln^3(z)}{3!} - 2\eta(4)\ln(z). \tag{1.98}$$

This pattern gives (1.94) and (1.95), which can be written in one identity:

$$\operatorname{Li}_a(-z) + (-1)^a \operatorname{Li}_a\left(-\frac{1}{z}\right) = -2\sum_{k=0}^{\lfloor \frac{a}{2} \rfloor} \frac{\eta(2k)}{(a-2k)!} \ln^{a-2k}(z). \tag{1.99}$$

By writing the integral forms of $\operatorname{Li}_{2a+1}(-z)$ and $\operatorname{Li}_{2a+1}(-\frac{1}{z})$, we have

$$\begin{aligned}
&\operatorname{Li}_{2a+1}(-z) - \operatorname{Li}_{2a+1}\left(-\frac{1}{z}\right) \\
&= \frac{1}{(2a)!}\int_0^1 \frac{-z\ln^{2a}(y)}{1+zy}\mathrm{d}y - \frac{1}{(2a)!}\int_0^1 \frac{-\frac{1}{z}\ln^{2a}(y)}{1+\frac{y}{z}}\mathrm{d}y \\
&= \frac{1}{(2a)!}\int_0^1 \ln^{2a-1}(y)\left(\frac{-z}{1+zy} + \frac{1}{z+y}\right)\mathrm{d}y \\
&= \frac{1}{(2a)!}\int_0^1 \frac{(1-z^2)\ln^{2a}(y)}{(1+zy)(z+y)}\mathrm{d}y.
\end{aligned}$$

So we have

$$\int_0^1 \frac{(1-z^2)\ln^{2a}(y)}{(1+zy)(z+y)}\mathrm{d}y = -2(2a)!\sum_{k=0}^{a}\frac{\eta(2a-2k)}{(2k+1)!}\ln^{2k+1}(z). \tag{1.100}$$

Similarly,

$$\begin{aligned}
&\operatorname{Li}_{2a}(-z)+\operatorname{Li}_{2a}\left(-\frac{1}{z}\right)\\
&= \frac{-1}{(2a-1)!}\int_0^1 \frac{-z\ln^{2a-1}(y)}{1+zy}\mathrm{d}y - \frac{1}{(2a-1)!}\int_0^1 \frac{-\frac{1}{z}\ln^{2a-1}(y)}{1+\frac{y}{z}}\mathrm{d}y\\
&= \frac{1}{(2a-1)!}\int_0^1 \ln^{2a-1}(y)\left(\frac{z}{1+zy}+\frac{1}{z+y}\right)\mathrm{d}y\\
&= \frac{1}{(2a-1)!}\int_0^1 \frac{(1+2zy+z^2)\ln^{2a-1}(y)}{(1+zy)(z+y)}\mathrm{d}y.
\end{aligned}$$

Therefore,

$$\int_0^1 \frac{(1+2zy+z^2)\ln^{2a-1}(y)}{(1+zy)(z+y)}\mathrm{d}y = -2(2a-1)!\sum_{k=0}^{a}\frac{\eta(2a-2k)}{(2k)!}\ln^{2k}(z). \tag{1.101}$$

Let's put $z=-1+i$ in (1.96) then consider the real parts of both sides,

$$\Re\left\{\operatorname{Li}_3(1-i)-\operatorname{Li}_3\left(\frac{1+i}{2}\right)\right\} = \Re\left\{-\zeta(2)\ln(-1+i)-\frac{1}{6}\ln^3(-1+i)\right\}.$$

Substitute the values (1.27) and (1.93),

$$\Re\operatorname{Li}_3\left(\frac{1+i}{2}\right) = \frac{\ln^3(2)}{48}-\frac{\ln(2)}{32}\zeta(2)+\frac{35}{64}\zeta(3) = \Re\operatorname{Li}_3\left(\frac{1-i}{2}\right). \tag{1.102}$$

Similarly,

$$\begin{aligned}
\Im\left\{\operatorname{Li}_3(1-i)-\operatorname{Li}_3\left(\frac{1+i}{2}\right)\right\} &= \Im\left\{-\zeta(2)\ln(-1+i)-\frac{1}{6}\ln^3(-1+i)\right\}\\
&= -\frac{7\pi^3}{128}-\frac{3\pi}{32}\ln^2(2). \qquad (1.103)
\end{aligned}$$

1.16.10 Second Polylogarithm Inversion Formula

Let $\Re(z) \leq 1$. Then the following two formulas hold:

$$\operatorname{Li}_{2a}(z) + \operatorname{Li}_{2a}\left(\frac{1}{z}\right) = \frac{i\pi \ln^{2a-1}(z)}{(2a-1)!} + 2\sum_{k=0}^{a} \frac{\zeta(2a-2k)}{(2k)!} \ln^{2k}(z); \quad (1.104)$$

$$\operatorname{Li}_{2a+1}(z) - \operatorname{Li}_{2a+1}\left(\frac{1}{z}\right) = \frac{i\pi \ln^{2a}(z)}{(2a)!} + 2\sum_{k=0}^{a} \frac{\zeta(2a-2k)}{(2k+1)!} \ln^{2k+1}(z). \quad (1.105)$$

Proof.

$$\begin{aligned}
\operatorname{Li}_2(z) + \operatorname{Li}_2\left(\frac{1}{z}\right) &= \int \mathrm{d}\left(\operatorname{Li}_2(z) + \operatorname{Li}_2\left(\frac{1}{z}\right)\right) \\
&= \int \left(-\frac{\ln(1-z)}{z} + \frac{\ln\left(\frac{z-1}{z}\right)}{z}\right) \mathrm{d}z \\
&= -\int \frac{\ln(-z)}{z} \mathrm{d}z \\
&\quad \{\text{write } (-1) = i\pi\} \\
&= -\int \frac{i\pi + \ln(z)}{z} \mathrm{d}z \\
&= -i\pi \ln(z) - \frac{1}{2}\ln^2(z) + c.
\end{aligned}$$

Setting $z = 1$ gives $c = 2\operatorname{Li}_2(1) = 2\zeta(2)$. Therefore

$$\operatorname{Li}_2(z) + \operatorname{Li}_2\left(\frac{1}{z}\right) = -i\pi \ln(z) - \frac{1}{2}\ln^2(z) + 2\zeta(2). \quad (1.106)$$

Keep integrating (1.106) from $z = 1$ to z after dividing by z, we get

$$\operatorname{Li}_3(z) - \operatorname{Li}_3\left(\frac{1}{z}\right) = -\frac{i\pi \ln^2(z)}{2} - \frac{\ln^3(z)}{3!} + 2\zeta(2)\ln(z); \quad (1.107)$$

$$\operatorname{Li}_4(z) + \operatorname{Li}_4\left(\frac{1}{z}\right) = -\frac{i\pi \ln^3(z)}{3!} - \frac{\ln^4(z)}{4!} + 2\zeta(2)\frac{\ln^2(z)}{2} + 2\zeta(4); \quad (1.108)$$

$$\operatorname{Li}_5(z) - \operatorname{Li}_5\left(\frac{1}{z}\right) = -\frac{i\pi \ln^4(z)}{4!} - \frac{\ln^5(z)}{5!} + 2\zeta(2)\frac{\ln^3(z)}{3!} + 2\zeta(4)\ln(z). \quad (1.109)$$

This pattern shows

$$\operatorname{Li}_{2a}(z)+\operatorname{Li}_{2a}\left(\frac{1}{z}\right)=-\frac{i\pi\ln^{2a-1}(z)}{(2a-1)!}+2\sum_{k=0}^{a}\frac{\zeta(2a-2k)}{(2k)!}\ln^{2k}(z);$$

$$\operatorname{Li}_{2a+1}(z)-\operatorname{Li}_{2a+1}\left(\frac{1}{z}\right)=-\frac{i\pi\ln^{2a}(z)}{(2a)!}+2\sum_{k=0}^{a}\frac{\zeta(2a-2k)}{(2k+1)!}\ln^{2k+1}(z).$$

The last two identities are defined for $\Re(z)\geq 1$ and this fact explains why their imaginary parts are negatives unlike those in (1.104) and (1.105) .

To finish the proof, let $z\to\frac{1}{z}$.

By writing the integral forms of $\operatorname{Li}_{2a}(z)$ and $\operatorname{Li}_{2a}(\frac{1}{z})$, we have

$$\begin{aligned}&\operatorname{Li}_{2a}(z)+\operatorname{Li}_{2a}\left(\frac{1}{z}\right)\\&=\frac{-1}{(2a-1)!}\int_0^1\frac{z\ln^{2a-1}(y)}{1-zy}\mathrm{d}y-\frac{-1}{(2a-1)!}\int_0^1\frac{\frac{1}{z}\ln^{2a-1}(y)}{1-\frac{y}{z}}\mathrm{d}y\\&=\frac{-1}{(2a-1)!}\int_0^1\ln^{2a-1}(y)\left(\frac{z}{1-zy}+\frac{1}{z-y}\right)\mathrm{d}y\\&=\frac{-1}{(2a-1)!}\int_0^1\frac{(1-2zy+z^2)\ln^{2a-1}(y)}{(1-zy)(z-y)}\mathrm{d}y.\end{aligned}$$

Thus,

$$\begin{aligned}\int_0^1\frac{(1-2zy+z^2)\ln^{2a-1}(y)}{(1-zy)(z-y)}\mathrm{d}y&=-i\pi\ln^{2a-1}(z)\\&-2(2a-1)!\sum_{k=0}^{a}\frac{\zeta(2a-2k)}{(2k)!}\ln^{2k}(z).\end{aligned}\tag{1.110}$$

Similarly,

$$\begin{aligned}&\operatorname{Li}_{2a+1}(z)-\operatorname{Li}_{2a+1}\left(\frac{1}{z}\right)\\&=\frac{-1}{(2a)!}\int_0^1\frac{z\ln^{2a}(y)}{1-zy}\mathrm{d}y-\frac{-1}{(2a)!}\int_0^1\frac{\frac{1}{z}\ln^{2a}(y)}{1-\frac{y}{z}}\mathrm{d}y\\&=\frac{-1}{(2a)!}\int_0^1\ln^{2a}(y)\left(\frac{z}{1-zy}+\frac{1}{z-y}\right)\mathrm{d}y\\&=\frac{-1}{(2a)!}\int_0^1\frac{(1-z^2)\ln^{2a}(y)}{(1-zy)(z-y)}\mathrm{d}y.\end{aligned}$$

Thus,

$$\int_0^1 \frac{(1-z^2)\ln^{2a}(y)}{(1-zy)(z-y)}\mathrm{d}y = -i\pi\ln^{2a}(z) - 2(2a)!\sum_{k=0}^{a}\frac{\zeta(2a-2k)}{(2k+1)!}\ln^{2k+1}(z). \tag{1.111}$$

Setting $z = 2$ in (1.106), (1.107), (1.108), and (1.109), since they are defined for $z \geq 1$, yields:

$$\mathrm{Li}_2(2) = \frac{3}{2}\zeta(2) - \pi\ln(2)i\,; \tag{1.112}$$

$$\mathrm{Li}_3(2) = \frac{7}{8}\zeta(3) + \frac{3}{2}\ln(2)\zeta(2) - \frac{\pi}{2}\ln^2(2)i\,; \tag{1.113}$$

$$\mathrm{Li}_4(2) = -\,\mathrm{Li}_4\left(\frac{1}{2}\right) + 2\zeta(4) + \ln^2(2)\zeta(2) - \frac{1}{24}\ln^4(2) - \frac{\pi}{6}\ln^3(2)i\,; \tag{1.114}$$

$$\mathrm{Li}_5(2) = \mathrm{Li}_5\left(\frac{1}{2}\right) + 2\ln(2)\zeta(4) + \frac{1}{3}\ln^3(2)\zeta(2) - \frac{1}{120}\ln^5(2) - \frac{\pi}{24}\ln^4(2)i. \tag{1.115}$$

The values of $\mathrm{Li}_2(\frac{1}{2})$ and $\mathrm{Li}_3(\frac{1}{2})$, given in (1.81) and (1.92), were used in (1.112) and (1.113). These four results can also be obtained by setting $z = \frac{1}{2}$ in (1.104) and (1.105).

For more results of polylogarithm functions, check [15].

1.17 Harmonic Number

1.17.1 Definition

The n-th generalized harmonic number of order a is defined by

$$H_n^{(a)} = \sum_{k=1}^{n}\frac{1}{k^a} = 1 + \frac{1}{2^a} + \frac{1}{3^a} + \cdots + \frac{1}{n^a}. \tag{1.116}$$

One of the properties of the harmonic number is

$$H_n^{(a)} - H_{n-1}^{(a)} = \frac{1}{n^a}, \tag{1.117}$$

which can be proved by taking the difference of

$$H_n^{(a)} = 1 + \frac{1}{2^a} + \frac{1}{3^a} + \cdots + \frac{1}{(n-1)^a} + \frac{1}{n^a}$$

and

$$H_{n-1}^{(a)} = 1 + \frac{1}{2^a} + \frac{1}{3^a} + \cdots + \frac{1}{(n-1)^a}.$$

With $a = 1$ in (1.116) and (1.117), we have:

$$H_n^{(1)} = H_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, \tag{1.118}$$

$$H_n - H_{n-1} = \frac{1}{n}. \tag{1.119}$$

Another property is

$$H_{2n}^{(a)} - 2^{-a} H_n^{(a)} = \sum_{k=1}^{n} \frac{1}{(2k-1)^a}. \tag{1.120}$$

To show that, begin with the RHS:

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{(2k-1)^a} &= 1 + \frac{1}{3^a} + \frac{1}{5^a} + \cdots + \frac{1}{(2n-1)^a} \\
&\left\{\text{add and subtract } \frac{1}{2^a} + \frac{1}{4^a} + \frac{1}{6^a} + \cdots + \frac{1}{(2n)^a}\right\} \\
&= 1 + \frac{1}{2^a} + \frac{1}{3^a} + \cdots + \frac{1}{(2n)^a} - \left(\frac{1}{2^a} + \frac{1}{4^a} + \frac{1}{6^a} + \cdots + \frac{1}{(2n)^a}\right) \\
&= 1 + \frac{1}{2^a} + \frac{1}{3^a} + \cdots + \frac{1}{(2n)^a} - \frac{1}{2^a}\left(1 + \frac{1}{2^a} + \frac{1}{3^a} \cdots + \frac{1}{n^a}\right) \\
&= \sum_{k=1}^{2n} \frac{1}{k^a} - \frac{1}{2^a} \sum_{k=1}^{n} \frac{1}{k^a} \\
&= H_{2n}^{(a)} - 2^{-a} H_n^{(a)}.
\end{aligned}$$

Let's define

$$f_n = H_{2n}^{(a)} - 2^{-a} H_n^{(a)} = \sum_{k=1}^{n} \frac{1}{(2k-1)^a},$$

we have

$$\begin{aligned}
f_{n+1} - f_n &= \sum_{k=1}^{n+1} \frac{1}{(2k-1)^a} - \sum_{k=1}^{n} \frac{1}{(2k-1)^a} \\
&\left\{\text{use } \sum_{k=1}^{n+1} f(k) - \sum_{k=1}^{n} f(k) = f(n+1)\right\}
\end{aligned}$$

$$= \frac{1}{(2n+1)^a}. \tag{1.121}$$

1.17.2 Integral Representation

The integral form of $H_n^{(a)}$ is defined by

$$H_n^{(a)} = \frac{(-1)^{a-1}}{(a-1)!}\int_0^1 \frac{\ln^{a-1}(x)(1-x^n)}{1-x}\mathrm{d}x, \quad \Re(n) > -1. \tag{1.122}$$

Proof. Using (1.36):

$$\frac{1}{k^a} = \frac{(-1)^{a-1}}{(a-1)!}\int_0^1 x^{k-1}\ln^{a-1}(x)\mathrm{d}x,$$

we have

$$\begin{aligned}
H_n^{(a)} &= \sum_{k=1}^{n}\frac{1}{k^a} = \sum_{k=1}^{n}\left(\frac{(-1)^{a-1}}{(a-1)!}\int_0^1 x^{k-1}\ln^{a-1}(x)\mathrm{d}x\right)\\
&\qquad \{\text{switch integration and summation}\}\\
&= \frac{(-1)^{a-1}}{(a-1)!}\int_0^1 \ln^{a-1}(x)\left(\sum_{k=1}^{n} x^{k-1}\right)\mathrm{d}x\\
&\qquad \{\text{use the geometric series}\}\\
&= \frac{(-1)^{a-1}}{(a-1)!}\int_0^1 \ln^{a-1}(x)\left(\frac{1-x^n}{1-x}\right)\mathrm{d}x.
\end{aligned}$$

This proof is valid if n is a positive integer, since we employed the finite geometric series. To justify the extension $\Re(n) > -1$, force integration by parts then use the derivatives of the beta function.

Note that setting $a = 1$ in (1.122) gives

$$H_n = \int_0^1 \frac{1-x^n}{1-x}\mathrm{d}x, \quad \Re(n) > -1. \tag{1.123}$$

Integrating this integral by parts yields

$$\begin{aligned}
\int_0^1 \frac{1-x^n}{1-x}\mathrm{d}x &= -\ln(1-x)(1-x^n)\Big|_0^1 - n\int_0^1 x^{n-1}\ln(1-x)\mathrm{d}x\\
&= 0 - n\int_0^1 x^{n-1}\ln(1-x)\mathrm{d}x
\end{aligned}$$

or

$$\int_0^1 x^{n-1}\ln(1-x)\mathrm{d}x = -\frac{H_n}{n}. \tag{1.124}$$

Let's break up the integrand in (1.122),

$$\begin{aligned} H_n^{(a)} &= \frac{(-1)^{a-1}}{(a-1)!}\int_0^1 \frac{\ln^{a-1}(x)}{1-x}\mathrm{d}x - \frac{(-1)^{a-1}}{(a-1)!}\int_0^1 \frac{x^n\ln^{a-1}(x)}{1-x}\mathrm{d}x \\ &\qquad \{\text{substitute the result from (3.8) for the first integral}\} \\ &= \zeta(a) - \frac{(-1)^{a-1}}{(a-1)!}\int_0^1 \frac{x^n\ln^{a-1}(x)}{1-x}\mathrm{d}x. \end{aligned} \tag{1.125}$$

Reorder the terms and replace a with $a+1$,

$$\int_0^1 \frac{x^n\ln^a(x)}{1-x}\mathrm{d}x = (-1)^a a!\left(\zeta(a+1) - H_n^{(a+1)}\right), \quad a\in\mathbb{Z}^+. \tag{1.126}$$

1.17.3 Infinite Series Representation

Another form of $H_n^{(a)}$ is

$$H_n^{(a)} = \sum_{k=1}^{\infty}\left(\frac{1}{k^a} - \frac{1}{(k+n)^a}\right), \quad n\notin\mathbb{Z}^-. \tag{1.127}$$

Proof. Expand $\frac{1}{1-x}$ in series,

$$\begin{aligned} H_n^{(a)} &= \frac{(-1)^{a-1}}{(a-1)!}\int_0^1 \frac{\ln^{a-1}(x)(1-x^n)}{1-x}\mathrm{d}x \\ &= \frac{(-1)^{a-1}}{(a-1)!}\int_0^1 \ln^{a-1}(x)(1-x^n)\left(\sum_{k=1}^{\infty}x^{k-1}\right)\mathrm{d}x \\ &\qquad \{\text{switch integration and summation}\} \\ &= \frac{(-1)^{a-1}}{(a-1)!}\sum_{k=1}^{\infty}\left(\int_0^1 \ln^{a-1}(x)(x^{k-1}-x^{k+n-1})\mathrm{d}x\right) \\ &\qquad \{\text{make use of the result (1.36)}\} \\ &= \sum_{k=1}^{\infty}\left(\frac{1}{k^a} - \frac{1}{(k+n)^a}\right). \end{aligned}$$

We also have

$$\begin{aligned}\sum_{k=1}^{\infty}\left(\frac{1}{(k+m)^a}-\frac{1}{(k+n)^a}\right)&=\sum_{k=1}^{\infty}\left(\frac{1}{(k+m)^a}-\frac{1}{(k+n)^a}+\frac{1}{k^a}-\frac{1}{k^a}\right)\\&=\sum_{k=1}^{\infty}\left(\frac{1}{k^a}-\frac{1}{(k+n)^a}\right)-\sum_{k=1}^{\infty}\left(\frac{1}{k^a}-\frac{1}{(k+m)^a}\right)\\&=H_n^{(a)}-H_m^{(a)}.\end{aligned}\tag{1.128}$$

Taking $a=1$ in (1.127) and (1.128) gives

$$H_n=\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+n}\right)=\sum_{k=1}^{\infty}\frac{n}{k(k+n)},\tag{1.129}$$

$$H_n-H_m=\sum_{k=1}^{\infty}\left(\frac{1}{k+m}-\frac{1}{k+n}\right).\tag{1.130}$$

To find the derivative of the harmonic number, rewrite (1.127) as

$$H_n^{(a)}=\zeta(a)-\sum_{k=1}^{\infty}\frac{1}{(k+n)^a},\quad a\in\mathbb{Z}_{>1}.\tag{1.131}$$

Next, differentiate both sides with respect to n,

$$\begin{aligned}\frac{\partial}{\partial n}H_n^{(a)}&=0+a\sum_{k=1}^{\infty}\frac{1}{(k+n)^{a+1}}\\&=a\left(\zeta(a+1)-H_n^{(a+1)}\right).\end{aligned}\tag{1.132}$$

1.18 Skew Harmonic Number

1.18.1 Definition

The n-th generalized skew harmonic number of order a is defined by

$$\overline{H}_n^{(a)}=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^a}=1-\frac{1}{2^a}+\frac{1}{3^a}-\cdots+\frac{(-1)^{n-1}}{n^a}.\tag{1.133}$$

One of the properties of the skew harmonic number is

$$\overline{H}_n^{(a)}-\overline{H}_{n-1}^{(a)}=\frac{(-1)^{n-1}}{n^a}.\tag{1.134}$$

To show that, take the difference of

$$\overline{H}_n^{(a)} = 1 - \frac{1}{2^a} + \frac{1}{3^a} - \cdots + \frac{(-1)^n}{(n-1)^a} + \frac{(-1)^{n-1}}{n^a}$$

and

$$\overline{H}_{n-1}^{(a)} = 1 - \frac{1}{2^a} + \frac{1}{3^a} - \cdots + \frac{(-1)^n}{(n-1)^a}.$$

Taking $a = 1$ in (1.133) and (1.134) gives:

$$\overline{H}_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}, \tag{1.135}$$

$$\overline{H}_n - \overline{H}_{n-1} = \frac{(-1)^{n-1}}{n}. \tag{1.136}$$

Another property is

$$\overline{H}_{2n} = H_{2n} - H_n. \tag{1.137}$$

To show that, we begin with the definition of $\overline{H}_{2n}$:

$$\begin{aligned}
\overline{H}_{2n} &= \sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2n-1} - \frac{1}{2n} \\
&\qquad \{\text{split the terms into odd and even}\} \\
&= 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1} - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n}\right) \\
&\qquad \left\{\text{add and subtract } \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n}\right\} \\
&= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n-1} + \frac{1}{2n} - 2\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n}\right) \\
&= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n-1} + \frac{1}{2n} - \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) \\
&= \sum_{k=1}^{2n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k} \\
&= H_{2n} - H_n.
\end{aligned}$$

A similar property is

$$\overline{H}_{2n+1} = H_{2n+1} - H_n, \tag{1.138}$$

which can be proved by using the definition of $\overline{H}_{2n+1}$:

$$\begin{aligned}
\overline{H}_{2n+1} &= \sum_{k=1}^{2n+1} \frac{(-1)^{k-1}}{k} \\
&\left\{\text{use } \sum_{k=1}^{n+1} f(k) = \sum_{k=1}^{n} f(k) + f(n+1)\right\} \\
&= \sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k} + \frac{1}{2n+1} \\
&= \overline{H}_{2n} + \frac{1}{2n+1} \\
&\{\text{substitute } (1.137)\} \\
&= H_{2n} - H_n + \frac{1}{2n+1} \\
&\left\{\text{use } H_{2n} + \frac{1}{2n+1} = H_{2n+1}\right\} \\
&= H_{2n+1} - H_n.
\end{aligned}$$

1.18.2 Integral Representation

The integral form of $\overline{H}_n^{(a)}$ is defined by

$$\overline{H}_n^{(a)} = \frac{(-1)^{a-1}}{(a-1)!} \int_0^1 \frac{\ln^{a-1}(x)(1-(-x)^n)}{1+x} \mathrm{d}x, \quad \Re(n) > -1. \tag{1.139}$$

Proof.

$$\begin{aligned}
\overline{H}_n^{(a)} = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^a} &= \sum_{k=1}^{n} (-1)^{k-1} \left(\frac{(-1)^{a-1}}{(a-1)!} \int_0^1 x^{k-1} \ln^{a-1}(x) \mathrm{d}x \right) \\
&= \frac{(-1)^{a-1}}{(a-1)!} \int_0^1 \ln^{a-1}(x) \left(\sum_{k=1}^{n} (-x)^{k-1} \right) \mathrm{d}x \\
&= \frac{(-1)^{a-1}}{(a-1)!} \int_0^1 \ln^{a-1}(x) \left(\frac{1-(-x)^n}{1+x} \right) \mathrm{d}x.
\end{aligned}$$

This proof is valid if n is a positive integer, since we employed the finite geometric series. To justify the extension $\Re(n) > -1$, force integration by parts then use the derivatives of the beta function.

With $a = 1$, we have

$$\begin{aligned}\overline{H}_n &= \int_0^1 \frac{1-(-x)^n}{1+x}\mathrm{d}x \\ &= \int_0^1 \frac{\mathrm{d}x}{1+x} - \int_0^1 \frac{(-x)^n}{1+x}\mathrm{d}x \\ &= \ln(2) - \int_0^1 \frac{(-x)^n}{1+x}\mathrm{d}x. \qquad (1.140)\end{aligned}$$

1.18.3 Infinite Series Representation

Another form of $\overline{H}_n^{(a)}$ is

$$\overline{H}_n^{(a)} = \sum_{k=1}^{\infty}\left(\frac{(-1)^{k-1}}{k^a} + \frac{(-1)^{k+n}}{(k+n)^a}\right), \quad n \notin \mathbb{Z}^-. \qquad (1.141)$$

Proof. Expand $\frac{1}{1+x}$ in series,

$$\begin{aligned}\overline{H}_n^{(a)} &= \frac{(-1)^{a-1}}{(a-1)!}\int_0^1 \frac{\ln^{a-1}(x)(1-(-x)^n)}{1+x}\mathrm{d}x \\ &= \frac{(-1)^{a-1}}{(a-1)!}\int_0^1 \ln^{a-1}(x)(1-(-x)^n)\left(\sum_{k=1}^{\infty}(-x)^{k-1}\right)\mathrm{d}x \\ &= \frac{(-1)^{a-1}}{(a-1)!}\sum_{k=1}^{\infty}\left(\int_0^1 \ln^{a-1}(x)((-x)^{k-1} - (-x)^{k+n-1})\mathrm{d}x\right) \\ &\qquad \{\text{make use of the result in (1.36)}\} \\ &= \sum_{k=1}^{\infty}\left(\frac{(-1)^{k-1}}{k^a} + \frac{(-1)^{k+n}}{(k+n)^a}\right).\end{aligned}$$

Setting $a = 1$ in (1.141) gives

$$\begin{aligned}\overline{H}_n &= \sum_{k=1}^{\infty}\left(\frac{(-1)^{k-1}}{k} + \frac{(-1)^{k+n}}{k+n}\right) \\ &= \ln(2) + \sum_{k=1}^{\infty}\frac{(-1)^{k+n}}{k+n}. \qquad (1.142)\end{aligned}$$

Multiplying both sides of by $(-1)^n$ gives

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k+n} = (-1)^n \left(\overline{H}_n - \ln(2)\right). \tag{1.143}$$

1.19 Digamma Function

1.19.1 Definition

The digamma function is defined as

$$\psi(n) = \frac{d}{dn} \ln(\Gamma(n)) = \frac{\Gamma'(n)}{\Gamma(n)}. \tag{1.144}$$

1.19.2 Digamma Reflection Formula

One of the key properties of the digamma function is

$$\psi(n) - \psi(1-n) = -\pi \cot(\pi n), \quad n \notin \mathbb{Z}. \tag{1.145}$$

Proof. Using $\frac{d}{dn} \ln(\Gamma(1-n)) = -\psi(1-n)$, we have

$$\begin{aligned}
\psi(n) - \psi(1-n) &= \frac{d}{dn} \ln(\Gamma(n)) + \frac{d}{dn} \ln(\Gamma(1-n)) \\
&= \frac{d}{dn} \ln(\Gamma(n)\Gamma(1-n)) \\
&\{\text{recall Euler's reflection formula (1.40)}\} \\
&= \frac{d}{dn} \ln(\pi \csc(\pi n)) \\
&= -\frac{\pi \csc(\pi n) \cot(\pi n)}{\csc(\pi n)} \\
&= -\pi \cot(\pi n).
\end{aligned}$$

1.19.3 Digamma–Harmonic Number Identity

Let $n \notin \mathbb{Z}^-$. Then the following identity holds:

$$\psi(n+1) = H_n - \gamma. \tag{1.146}$$

Proof.

$$\begin{aligned}
\psi(n+1) &= \frac{d}{dn}\ln(\Gamma(n+1)) \\
&\{\text{use } n\Gamma(n) = \Gamma(n+1) \text{ in the identity (1.39)}\} \\
&= \frac{d}{dn}\ln\prod_{k=1}^{\infty}\frac{\left(1+\frac{1}{k}\right)^n}{\left(1+\frac{n}{k}\right)} \\
&\left\{\text{use } \ln\prod_{k=1}^{\infty}a_k = \sum_{k=1}^{\infty}\ln(a_k) \text{ given in (1.15)}\right\} \\
&= \frac{d}{dn}\sum_{k=1}^{\infty}\ln\left(\frac{\left(1+\frac{1}{k}\right)^n}{\left(1+\frac{n}{k}\right)}\right) \\
&= \frac{d}{dn}\sum_{k=1}^{\infty}\left(n\ln\left(\frac{k+1}{k}\right) - \ln\left(\frac{k+n}{k}\right)\right) \\
&= \sum_{k=1}^{\infty}\left(\ln\left(\frac{k+1}{k}\right) - \frac{1}{k+n}\right) \\
&\{\text{add and subtract } 1/k \text{ then insert the limit}\} \\
&= \lim_{m\to\infty}\sum_{k=1}^{m}\left(\ln\left(\frac{k+1}{k}\right) - \frac{1}{k+n} + \frac{1}{k} - \frac{1}{k}\right) \\
&\{\text{rearrange the terms}\} \\
&= \lim_{m\to\infty}\left(\sum_{k=1}^{m}\ln\left(\frac{k+1}{k}\right) - \sum_{k=1}^{m}\frac{1}{k}\right) + \lim_{m\to\infty}\sum_{k=1}^{m}\left(\frac{1}{k} - \frac{1}{k+n}\right) \\
&\left\{\text{note that } \sum_{k=1}^{m}\frac{1}{k} = H_m \text{ and } \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+n}\right) = H_n\right\} \\
&= \lim_{m\to\infty}\left(\underbrace{\ln\left(\frac{2}{1}\right) + \ln\left(\frac{3}{2}\right) + \cdots + \ln\left(\frac{m+1}{m}\right)}_{\text{telescoping series}} - H_m\right) + H_n \\
&= \lim_{m\to\infty}\left(\ln(m+1) - H_m\right) + H_n
\end{aligned}$$

$$\begin{aligned}
&\{\text{let } m+1=n\ \} \\
&= \lim_{n\to\infty}(\ln(n)-H_{n-1})+H_n \\
&\left\{\text{write } H_{n-1}=H_n-\frac{1}{n} \text{ given in (1.119)}\right\} \\
&= -\lim_{n\to\infty}(H_n-\ln(n))+\lim_{n\to\infty}\frac{1}{n}+H_n \\
&\{\text{the first limit is the Euler–Mascheroni constant (1.164)}\} \\
&= -\gamma+0+H_n.
\end{aligned}$$

The range of n can be extended to $\Re(n)>-1$, since the integral forms of H_n and $\psi(n+1)$ are defined for $\Re(n)>-1$.

Let's set $n=0$ in (1.146),

$$\psi(1)=-\gamma. \tag{1.147}$$

Now set $n=-\frac{1}{2}$,

$$\begin{aligned}
\psi\left(\frac{1}{2}\right)+\gamma &= H_{-\frac{1}{2}}=\int_0^1\frac{1-x^{-\frac{1}{2}}}{1-x}\mathrm{d}x \\
&\overset{\sqrt{x}=y}{=} -2\int_0^1\frac{\mathrm{d}y}{1+y}=-2\ln(2).
\end{aligned} \tag{1.148}$$

We also have

$$\begin{aligned}
\psi(n+1)-\psi(n) &= H_n-\gamma-(H_{n-1}-\gamma) \\
&= H_n-H_{n-1}=\frac{1}{n}.
\end{aligned} \tag{1.149}$$

1.20 Polygamma Function

1.20.1 Definition

The polygamma function is defined as

$$\psi^{(a)}(n)=\frac{d^a}{dn^a}\psi(n)=\frac{d^{a+1}}{dn^{a+1}}\ln(\Gamma(n)). \tag{1.150}$$

Observe that

$$\psi^{(0)}(n)=\psi(n).$$

1.20.2 Series Representation

Another form of the polygamma function is

$$\psi^{(a)}(n+1)=\sum_{k=1}^{\infty}\frac{(-1)^{a+1}a!}{(k+n)^{a+1}},\quad a\in\mathbb{Z}_{\geq 1},\ n\notin\mathbb{Z}^{-}. \tag{1.151}$$

Proof (i). Using the definition of the polygamma function, we get

$$\begin{aligned}
\psi^{(a)}(n+1)&=\frac{d^{a+1}}{dn^{a+1}}\ln(\Gamma(n+1))\\
&\{\text{write } n\Gamma(n)=\Gamma(n+1) \text{ in } (1.39)\}\\
&=\frac{d^{a+1}}{dn^{a+1}}\ln\prod_{k=1}^{\infty}\frac{\left(1+\frac{1}{k}\right)^n}{1+\frac{n}{k}}\\
&=\frac{d^{a+1}}{dn^{a+1}}\sum_{k=1}^{\infty}\ln\left(\frac{\left(1+\frac{1}{k}\right)^n}{1+\frac{n}{k}}\right)\\
&=\frac{d^{a+1}}{dn^{a+1}}\sum_{k=1}^{\infty}\left(n\ln\left(1+\frac{1}{k}\right)-\ln\left(\frac{k+n}{k}\right)\right)\\
&\left\{\text{use } \frac{d^q}{dx^q}\ln(x)=\frac{(-1)^{q-1}(q-1)!}{x^q} \text{ for the second sum}\right\}\\
&\{\text{and note that the first sum vanishes when } a\geq 1\}\\
&=0-\sum_{k=1}^{\infty}\frac{(-1)^a a!}{(k+n)^{a+1}}.
\end{aligned}$$

Proof (ii). Differentiate both sides of (1.146):

$$\psi(n+1)=H_n-\gamma$$

a times with respect to n using the fact $\frac{d^a}{dn^a}\psi(n)=\psi^{(a)}(n)$,

$$\begin{aligned}
\psi^{(a)}(n+1)&=\frac{d^a}{dn^a}(H_n-\gamma)\\
&\{\text{recall the series representation } (1.129)\}\\
&=\frac{d^a}{dn^a}\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+n}\right)-0\\
&\left\{\text{use } \frac{d^a}{dx^a}\frac{1}{x}=\frac{(-1)^a a!}{x^{a+1}}\right\}
\end{aligned}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{a+1} a!}{(k+n)^{a+1}}.$$

If we replace n with $n-1$ in (1.151) then shift the index by $+1$, we get

$$\psi^{(a)}(n) = \sum_{k=0}^{\infty} \frac{(-1)^{a+1} a!}{(k+n)^{a+1}}, \quad n \notin \mathbb{Z}_{\leq 0}. \tag{1.152}$$

1.20.3 Integral Representation

The integral form of the polygamma function is

$$\psi^{(a)}(n+1) = -\int_0^1 \frac{x^n \ln^a(x)}{1-x} \mathrm{d}x, \quad \Re(n) > -1. \tag{1.153}$$

Proof (i).

$$\begin{aligned}
\psi^{(a)}(n+1) &= -\sum_{k=1}^{\infty} \frac{(-1)^a a!}{(k+n)^{a+1}} \\
&= -\sum_{k=1}^{\infty} \left(\int_0^1 x^{k+n-1} \ln^a(x) \mathrm{d}x \right) \\
&= -\int_0^1 x^n \ln^a(x) \left(\sum_{k=1}^{\infty} x^{k-1} \right) \mathrm{d}x \\
&= -\int_0^1 x^n \ln^a(x) \left(\frac{1}{1-x} \right) \mathrm{d}x.
\end{aligned}$$

This proof is valid if $n \notin \mathbb{Z}^-$, since we used the definition $\sum_{k=1}^{\infty} \frac{1}{(k+n)^{a+1}}$. The extension $\Re(n) > -1$ can be justified by applying integration by parts then using the derivatives of the beta function.

Proof (ii). Differentiating both sides of (1.146) a times with respect to n,

$$\psi^{(a)}(n+1) = \frac{d^a}{dn^a}(H_n - \gamma)$$

{recall the integral form (1.123)}

$$= \frac{d^a}{dn^a} \int_0^1 \frac{1-x^n}{1-x} \mathrm{d}x$$

{use differentiation under the integral sign theorem (2.64)}

$$= \int_0^1 \frac{\partial^a}{\partial n^a} \frac{1-x^n}{1-x} \mathrm{d}x$$

$$\left\{\text{use } \frac{\partial^a}{\partial n^a} x^n = \ln^a(x) x^n\right\}$$

$$= -\int_0^1 \frac{x^n \ln^a(x)}{1-x} \mathrm{d}x.$$

By the way, comparing (1.126) with (1.153) yields

$$\psi^{(a)}(n+1) = (-1)^a a! \left(H_n^{(a+1)} - \zeta(a+1)\right), \quad \Re(n) > -1. \tag{1.154}$$

1.20.4 Evaluation of $\psi^{(a)}(1)$

Let $a \in \mathbb{Z}^+$. Then we have

$$\psi^{(a)}(1) = (-1)^{a-1} a! \, \zeta(a+1). \tag{1.155}$$

Proof. Put $n = 0$ in (1.151),

$$\psi^{(a)}(1) = \sum_{k=1}^{\infty} \frac{(-1)^{a-1} a!}{k^{a+1}} = (-1)^{a-1} a! \, \zeta(a+1).$$

Examples:

$$\psi^{(1)}(1) = \zeta(2);$$

$$\psi^{(2)}(1) = -2\zeta(3);$$

$$\psi^{(3)}(1) = 6\zeta(4);$$

$$\psi^{(4)}(1) = -24\zeta(5).$$

By the way, using the definition of the Taylor series:

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[\frac{d^n}{dx^n} f(x)\right]_{x=0} x^n,$$

we have

$$\psi(1-x) + \gamma = \sum_{n=0}^{\infty} \frac{1}{n!} \left[\frac{d^n(\psi(1-x)+\gamma)}{dx^n}\right]_{x=0} x^n$$

$$\{\text{separate the first term using } \psi(1) = -\gamma\}$$

$$= 0 + \sum_{n=1}^{\infty} \frac{1}{n!} \left[\frac{d^n(\psi(1-x)+\gamma)}{dx^n} \right]_{x=0} x^n$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!} \left[(-1)^n \psi^{(n)}(1-x) \right]_{x=0} x^n$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{\psi^{(n)}(1)}{n!} x^n$$

$$\{\text{substitute the value of } \psi^{(a)}(1)\}$$

$$= -\sum_{n=1}^{\infty} \zeta(n+1)\, x^n$$

$$\{\text{shift the index}\}$$

$$= -\sum_{n=0}^{\infty} \zeta(n+2)\, x^{n+1}$$

or

$$\frac{\psi(1-x)+\gamma}{x} = -\sum_{n=0}^{\infty} \zeta(n+2) x^n, \quad |x| < 1. \tag{1.156}$$

1.20.5 Evaluation of $\psi^{(a)}\left(\frac{1}{2}\right)$

Let $a \in \mathbb{Z}^+$. Then we have

$$\psi^{(a)}\left(\frac{1}{2}\right) = (-1)^a a!\, (1 - 2^{a+1}) \zeta(a+1). \tag{1.157}$$

Proof. Put $n = \frac{1}{2}$ in (1.152),

$$\psi^{(a)}\left(\frac{1}{2}\right) = \sum_{k=0}^{\infty} \frac{(-1)^{a+1} a!}{(k+1/2)^{a+1}}$$

$$= (-1)^{a-1} a!\, 2^{a+1} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{a+1}}.$$

This sum is given in (1.65) and the proof is finished.

Examples:

$$\psi^{(1)}\left(\frac{1}{2}\right) = 3\zeta(2);$$

$$\psi^{(2)}\left(\frac{1}{2}\right) = -14\zeta(3);$$
$$\psi^{(3)}\left(\frac{1}{2}\right) = 90\zeta(4);$$
$$\psi^{(4)}\left(\frac{1}{2}\right) = -744\zeta(5).$$

1.20.6 Evaluation of $\psi^{(2a)}\left(\frac{1}{4}\right)$

Let $a \in \mathbb{Z}^+$. Then we have

$$\psi^{(2a)}\left(\frac{1}{4}\right) = (2^{2a} - 2^{1+4a})(2a)!\,\zeta(2a+1) - \frac{|E_{2a}|}{4}(2\pi)^{2a+1}. \tag{1.158}$$

Proof. Replace a with $2a+1$ in (1.63),

$$\beta(2a+1) = -\frac{2^{-1-4a}}{(2a)!}\psi^{(2a)}\left(\frac{1}{4}\right) - (1 - 2^{-1-2a})\zeta(2a+1). \tag{1.159}$$

Combine (1.66) and (1.159) to finish the proof.

Examples:

$$\psi^{(2)}\left(\frac{1}{4}\right) = -56\zeta(3) - 2\pi^3;$$
$$\psi^{(4)}\left(\frac{1}{4}\right) = -11904\zeta(5) - 40\pi^5;$$
$$\psi^{(6)}\left(\frac{1}{4}\right) = -5852160\zeta(7) - 1952\pi^7;$$
$$\psi^{(8)}\left(\frac{1}{4}\right) = -5274501120\zeta(9) - 177280\pi^9.$$

1.20.7 Evaluation of $\psi^{(2a)}\left(\frac{3}{4}\right)$

Let $a \in \mathbb{Z}^+$. Then we have

$$\psi^{(2a)}\left(\frac{3}{4}\right) = \psi^{(2a)}\left(\frac{1}{4}\right) + \frac{|E_{2a}|}{2}(2\pi)^{2a+1}. \tag{1.160}$$

Proof. Differentiate both sides of (1.145):

$$\psi(1-n)-\psi(n)=\pi\cot(\pi n)$$

$2a$ times with respect to n then let $n \to \frac{1}{4}$,

$$\psi^{(2a)}\left(\frac{3}{4}\right)-\psi^{(2a)}\left(\frac{1}{4}\right)=\pi\lim_{n\to\frac{1}{4}}\frac{d^{2a}}{dn^{2a}}\cot(\pi n).$$

By employing the identity $\cot(x)=\csc(2x)+\cot(2x)$, we obtain

$$\begin{aligned}\lim_{n\to\frac{1}{4}}\frac{d^{2a}}{dn^{2a}}\cot(\pi n) &= \lim_{n\to\frac{1}{4}}\frac{d^{2a}}{dn^{2a}}\csc(2\pi n)+\lim_{n\to\frac{1}{4}}\frac{d^{2a}}{dn^{2a}}\cot(2\pi n)\\ &\{\text{let } n=m/2 \text{ and so} \to dn=dm/2\}\\ &=2^{2a}\lim_{m\to\frac{1}{2}}\frac{d^{2a}}{dm^{2a}}\csc(\pi m)+2^{2a}\lim_{m\to\frac{1}{2}}\frac{d^{2a}}{dm^{2a}}\cot(\pi m)\\ &\{\text{these two limits are given in (1.168) and (1.175)}\}\\ &=2^{2a}\,|E_{2a}|\pi^{2a}+0\\ &=|E_{2a}|(2\pi)^{2a}. \end{aligned}\tag{1.161}$$

Examples:

$$\begin{aligned}\psi^{(2)}\left(\frac{3}{4}\right)&=\psi^{(2)}\left(\frac{1}{4}\right)+4\pi^3;\\ \psi^{(4)}\left(\frac{3}{4}\right)&=\psi^{(4)}\left(\frac{1}{4}\right)+80\pi^5;\\ \psi^{(6)}\left(\frac{3}{4}\right)&=\psi^{(6)}\left(\frac{1}{4}\right)+3904\pi^7;\\ \psi^{(8)}\left(\frac{3}{4}\right)&=\psi^{(8)}\left(\frac{1}{4}\right)+354560\pi^9.\end{aligned}$$

1.21 Catalan's Constant

Catalan's constant, denoted by G, is defined as

$$G=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=1-\frac{1}{3^2}+\frac{1}{5^2}-\cdots\tag{1.162}$$

The Catalan's constant is a special case of the Dirichlet beta function (1.61):

$$\beta(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^z}.$$

In (1.63), we also see that

$$G = -\int_0^1 \frac{\ln(x)}{1+x^2}\mathrm{d}x = \frac{\psi^{(1)}\left(\frac{1}{4}\right)}{48} - \frac{3}{4}\zeta(2). \tag{1.163}$$

For more integral and series representations of the Catalan's constant, check [41].

1.22 Euler–Mascheroni Constant

The Euler–Mascheroni constant is defined as

$$\gamma = \lim_{n\to\infty} \left(H_n - \ln(n)\right). \tag{1.164}$$

The Euler–Mascheroni constant, denoted by γ, is defined as the area bounded by the two functions, $y = 1/x$ and $y = 1/\lfloor x \rfloor$, where $\lfloor x \rfloor$ is the floor function, over the interval $x \in [1, \infty)$. To get the form in (1.164), we calculate the bounded area over the interval $x \in [1, n]$ then we let integer $n \to \infty$:

$$\begin{aligned}
\gamma &= \lim_{n\to\infty} \left(\int_1^n \frac{\mathrm{d}x}{\lfloor x \rfloor} - \int_1^n \frac{\mathrm{d}x}{x}\right) \\
&\left\{\text{note that } \int_1^n \frac{\mathrm{d}x}{\lfloor x \rfloor} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} = H_{n-1}\right\} \\
&= \lim_{n\to\infty} \left(H_{n-1} - \int_1^n \frac{\mathrm{d}x}{x}\right) \\
&\left\{\text{we have } H_{n-1} = H_n - \frac{1}{n} \text{ and } \int_1^n \frac{\mathrm{d}x}{x} = \ln(n)\right\} \\
&= \lim_{n\to\infty} \left(H_n - \frac{1}{n} - \ln(n)\right) \\
&= \lim_{n\to\infty} \left(H_n - \ln(n)\right) - \lim_{n\to\infty} \frac{1}{n} \\
&= \lim_{n\to\infty} \left(H_n - \ln(n)\right) - 0.
\end{aligned}$$

For more representations of the Euler–Mascheroni constant, see [42].

1.23 Euler Numbers

The Euler numbers are defined as

$$E_{2n}=\sum_{k=1}^{2n}\sum_{j=0}^{2k}(-1)^{k+j}2^{-k}j^{n}\binom{2k}{j}(k-j)^{2n}. \tag{1.165}$$

Some values:

$$E_0=1,\ E_2=-1,\ E_4=5,\ E_6=-61,$$

while the odd-indexed Euler numbers are all zero.

The Euler numbers appear in the Taylor series of $\operatorname{sech} x$:

$$\operatorname{sech} x=1-\frac{x^2}{2!}+5\frac{x^4}{4!}-61\frac{x^6}{6!}+\cdots=\sum_{n=0}^{\infty}E_n\frac{x^n}{n!}.$$

Let's replace x with ix then consider the real parts of both sides, we have

$$\Re\left\{\operatorname{sech}(ix)\right\}=\Re\sum_{n=0}^{\infty}i^nE_n\frac{x^n}{n!}.$$

Using $\Re\left\{\operatorname{sech}(ix)\right\}=\sec x$ and $\Re\sum_{n=0}^{\infty}i^na_n=\sum_{n=0}^{\infty}(-1)^na_{2n}$, which follows from applying the same approach shown in (1.10), we have

$$\sec x=\sum_{n=0}^{\infty}\frac{(-1)^nE_{2n}}{(2n)!}x^{2n}=\sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n}. \tag{1.166}$$

Let's differentiate both sides $2a$ times with respect to x then let $x\to 0$

$$\begin{aligned}
\lim_{x\to 0}\frac{d^{2a}}{dx^{2a}}\sec(x)&=\lim_{x\to 0}\frac{d^{2a}}{dx^{2a}}\sum_{k=0}^{\infty}\frac{|E_{2k}|}{(2k)!}x^{2k}\\
&\left\{\text{use }\frac{d^q}{dx^q}x^n=\frac{n!}{(n-q)!}x^{n-q},\quad n\ge q\right\}\\
&=\lim_{x\to 0}\sum_{k=a}^{\infty}\frac{|E_{2k}|}{(2k-2a)!}x^{2k-2a}\\
&=\lim_{x\to 0}\left(|E_{2a}|+\frac{|E_{2a+2}|}{2}x^2+\frac{|E_{2a+4}|}{4}x^4+\cdots\right)\\
&=|E_{2a}|.
\end{aligned} \tag{1.167}$$

The following limit is related: Set $s = t + \frac{1}{2}$, we have

$$\begin{aligned}\lim_{s\to\frac{1}{2}} \frac{d^{2a}}{ds^{2a}} \csc(s\pi) &= \lim_{t\to 0} \frac{d^{2a}}{dt^{2a}} \csc\left(t\pi + \frac{\pi}{2}\right)\\ &= \lim_{t\to 0} \frac{d^{2a}}{dt^{2a}} \sec(t\pi)\\ &\{\text{let } t = x/\pi \text{ and so } dt = dx/\pi\}\\ &= \pi^{2a} \lim_{x\to 0} \frac{d^{2a}}{dx^{2a}} \sec(x)\\ &\{\text{substitute the limit (1.167)}\}\\ &= |E_{2a}|\pi^{2a}. \end{aligned} \tag{1.168}$$

1.24 Bernoulli Numbers

The Bernoulli numbers are defined as

$$B_n = \sum_{k=0}^{n}\sum_{j=0}^{k} \frac{(-1)^j j^n \binom{k}{j}}{k+1}. \tag{1.169}$$

Some values:

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_3 = 0,\ B_4 = -\frac{1}{30},\ B_5 = 0,\ B_6 = \frac{1}{40}.$$

Note that $B_{2n+1} = 0$ for $n \in Z^+$.
The Bernoulli numbers appear in the Taylor series of $\frac{x}{e^x-1}$:

$$\frac{x}{e^x - 1} = 1 - \frac{1}{2}\frac{x}{1!} + \frac{1}{6}\frac{x^2}{2!} - \frac{1}{30}\frac{x^4}{4!} + \frac{1}{40}\frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

Let's replace x with $2ix$ then consider the real parts of both sides, we have

$$\Re\left\{\frac{2ix}{e^{2ix} - 1}\right\} = \Re\sum_{n=0}^{\infty} i^n B_n \frac{(2x)^n}{n!}.$$

Using $\Re\left\{\frac{2ix}{e^{2ix}-1}\right\} = x\cot x$ and $\Re\sum_{n=0}^{\infty} i^n a_n = \sum_{n=0}^{\infty} (-1)^n a_{2n}$, we have

$$\cot x = \frac{1}{x}\sum_{n=0}^{\infty} \frac{(-1)^n B_{2n}}{(2n)!}(2x)^{2n}. \tag{1.170}$$

By employing $\tan x = \cot x - 2\cot(2x)$, we find

$$\begin{aligned} \tan x &= \cot x - 2\cot(2x) \\ &= \frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n B_{2n}}{(2n)!}(2x)^{2n} - 2\cdot\frac{1}{2x}\sum_{n=0}^{\infty}\frac{(-1)^n B_{2n}}{(2n)!}(4x)^{2n} \\ &= \frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n(1-2^{2n})B_{2n}}{(2n)!}(2x)^{2n}. \end{aligned} \tag{1.171}$$

Let's differentiate both sides $(2a-1)$ times with respect to x then let $x \to 0$

$$\begin{aligned} \lim_{x\to 0}\frac{d^{2a-1}}{dx^{2a-1}}\tan x &= \lim_{x\to\frac{1}{2}}\frac{d^{2a-1}}{dx^{2a-1}}\frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n(1-2^{2n})B_{2n}}{(2n)!}(2x)^{2n} \\ &= \lim_{x\to\frac{1}{2}}\frac{d^{2a-1}}{dx^{2a-1}}\sum_{n=0}^{\infty}\frac{(-1)^n(1-2^{2n})B_{2n}2^{2n}}{(2n)!}x^{2n-1} \\ &\quad \left\{\text{use } \frac{d^q}{dx^q}x^n = \frac{n!}{(n-q)!}x^{n-q}, \quad n \geq q\right\} \\ &= \lim_{x\to 0}\sum_{n=a}^{\infty}\frac{(-1)^n(1-2^{2n})B_{2n}2^{2n}(2n-1)!}{(2n)!(2n-2a)!}x^{2n-2a} \\ &= \lim_{x\to 0}\sum_{n=a}^{\infty}\underbrace{\frac{(-1)^n(1-2^{2n})B_{2n}2^{2n}}{(2n)(2n-2a)!}}_{f(n)}x^{2n-2a} \\ &= \lim_{x\to 0}\sum_{n=a}^{\infty}f(n)\,x^{2n-2a} \\ &= \lim_{x\to 0}\left(f(a) + f(a+1)\,x^2 + f(a+2)\,x^4 + \cdots\right) \\ &= f(a) \\ &= \frac{(-1)^a}{a}(1-2^{2a})B_{2a}2^{2a-1}. \end{aligned} \tag{1.172}$$

Applying the same approach using $\displaystyle\sum_{2n-1\geq 2a} f(n) = \sum_{n=a+1}^{\infty} f(n)$, we get

$$\begin{aligned} \lim_{x\to 0}\frac{d^{2a}}{dx^{2a}}\tan x &= \lim_{x\to 0}\sum_{n=a+1}^{\infty}\underbrace{\frac{(-1)^n(1-2^{2n})B_{2n}2^{2n}}{(2n)(2n-2a-1)!}}_{f(n)}x^{2n-2a-1} \\ &= \lim_{x\to 0}\sum_{n=a+1}^{\infty}f(n)\,x^{2n-2a-1} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{x\to 0}(f(a+1)\,x + f(a+2)\,x^3 + f(a+3)\,x^5 + \cdots) \\
&= 0. \qquad (1.173)
\end{aligned}
$$

The following two limits are related: Set $s = t + \frac{1}{2}$, we have

$$
\begin{aligned}
\lim_{s\to\frac{1}{2}} \frac{d^{2a-1}}{ds^{2a-1}}\cot(s\pi) &= \lim_{t\to 0}\frac{d^{2a-1}}{dt^{2a-1}}\cot\left(t\pi + \frac{\pi}{2}\right) \\
&= -\lim_{t\to 0}\frac{d^{2a-1}}{dt^{2a-1}}\tan(t\pi) \\
&\{\text{let } t = x/\pi \text{ and so } dt = dx/\pi\} \\
&= -\pi^{2a-1}\lim_{x\to 0}\frac{d^{2a-1}}{dx^{2a-1}}\tan(x) \\
&\{\text{plug in the limit (1.172)}\} \\
&= \frac{(-1)^a}{a}(2^{2a}-1)B_{2a}(2\pi)^{2a-1}. \qquad (1.174)
\end{aligned}
$$

$$
\begin{aligned}
\lim_{s\to\frac{1}{2}} \frac{d^{2a}}{ds^{2a}}\cot(s\pi) &= \lim_{t\to 0}\frac{d^{2a}}{dt^{2a}}\cot\left(t\pi + \frac{\pi}{2}\right) \\
&= -\lim_{t\to 0}\frac{d^{2a}}{dt^{2a}}\tan(t\pi) \\
&= -\pi^{2a}\lim_{x\to 0}\frac{d^{2a}}{dx^{2a}}\tan x \\
&\{\text{substitute the limit (1.173)}\} \\
&= 0. \qquad (1.175)
\end{aligned}
$$

Chapter 2

Generating Functions and Powerful Identities

Before we start deriving the generating functions, we need to prove the following series identity:

Let $\sum_{n=1}^{\infty} a_n x^n$ be a convergent series. Then the following identity holds:

$$\sum_{n=1}^{\infty} a_n x^n = \frac{1}{1-x} \sum_{n=1}^{\infty} (a_n - a_{n-1}) x^n, \quad a_0 = 0. \tag{2.1}$$

Proof. Multiply both sides of

$$1 = \frac{1}{1-x} - \frac{x}{1-x}$$

by $a_n x^n$ then take the summation over $n \geq 1$,

$$\begin{aligned}
\sum_{n=1}^{\infty} a_n x^n &= \frac{1}{1-x} \sum_{n=1}^{\infty} a_n x^n - \frac{1}{1-x} \sum_{n=1}^{\infty} a_n x^{n+1} \\
&\{\text{let the index } n \text{ of the second sum start from } 0, \text{ since } a_0 = 0\} \\
&= \frac{1}{1-x} \sum_{n=1}^{\infty} a_n x^n - \frac{1}{1-x} \sum_{n=0}^{\infty} a_n x^{n+1} \\
&\{\text{shift the index } n \text{ of the second sum by } -1\} \\
&= \frac{1}{1-x} \sum_{n=1}^{\infty} a_n x^n - \frac{1}{1-x} \sum_{n=1}^{\infty} a_{n-1} x^n
\end{aligned}$$

$$= \frac{1}{1-x}\sum_{n=1}^{\infty}(a_n - a_{n-1})x^n.$$

2.1 Generating Functions

2.1.1 $\sum_{n=1}^{\infty} H_n^{(a)} x^n$

Let $|x| < 1$. Then the following identity holds:

$$\sum_{n=1}^{\infty} H_n^{(a)} x^n = \frac{\operatorname{Li}_a(x)}{1-x}. \tag{2.2}$$

Proof. Since $H_0^{(a)} = 0$, it's valid to set $a_n = H_n^{(a)}$ in (2.1),

$$\begin{aligned}\sum_{n=1}^{\infty} H_n^{(a)} x^n &= \frac{1}{1-x}\sum_{n=1}^{\infty}\left(H_n^{(a)} - H_{n-1}^{(a)}\right)x^n\\ &\left\{\text{write } H_n^{(a)} - H_{n-1}^{(a)} = \frac{1}{n^a} \text{ given in (1.117)}\right\}\\ &= \frac{1}{1-x}\sum_{n=1}^{\infty}\frac{x^n}{n^a}\\ &= \frac{\operatorname{Li}_a(x)}{1-x}.\end{aligned}$$

Let the index n in (2.2) start from 0, since $H_0^{(a)} = 0$

$$\frac{\operatorname{Li}_a(x)}{1-x} = \sum_{n=0}^{\infty} H_n^{(a)} x^n = \sum_{n=1}^{\infty} H_{n-1}^{(a)} x^{n-1}. \tag{2.3}$$

The last step follows from shifting the index n by -1.
For a different proof, see [32, pp. 348–349].
If we set $a = 1$ in (2.2) and (2.3) using $\operatorname{Li}_1(x) = -\ln(1-x)$, we get

$$\sum_{n=1}^{\infty} H_n x^n = -\frac{\ln(1-x)}{1-x}, \quad |x| < 1, \tag{2.4}$$

$$\sum_{n=1}^{\infty} H_{n-1} x^{n-1} = -\frac{\ln(1-x)}{1-x}, \quad |x| < 1. \tag{2.5}$$

The result (2.4) is also proved in (2.60).

2.1.2 $\sum_{n=1}^{\infty} \frac{H_n}{n} x^n$

Let $-1 \leq x < 1$. Then the following identity holds:

$$\sum_{n=1}^{\infty} \frac{H_n}{n} x^n = \operatorname{Li}_2(x) + \frac{1}{2} \ln^2(1-x). \tag{2.6}$$

Proof (i).

$$\begin{aligned}
\sum_{n=1}^{\infty} H_n \frac{x^n}{n} &= \sum_{n=1}^{\infty} H_n \left(\int x^{n-1} \mathrm{d}x \right) \\
&= \int \frac{1}{x} \left(\sum_{n=1}^{\infty} H_n x^n \right) \mathrm{d}x \\
&\quad \{\text{recall the generating function (2.4)}\} \\
&= \int \frac{1}{x} \left(-\frac{\ln(1-x)}{1-x} \right) \mathrm{d}x \\
&= -\int \frac{\ln(1-x)}{x} \mathrm{d}x - \int \frac{\ln(1-x)}{1-x} \mathrm{d}x \\
&= \operatorname{Li}_2(x) + \frac{1}{2} \ln^2(1-x) + c.
\end{aligned}$$

Set $x = 0$, we get $c = 0$ and the proof is complete.

Proof (ii).

$$\begin{aligned}
\ln^2(1-x) &= (-\ln(1-x))(-\ln(1-x)) \\
&\quad \{\text{expand both logs in series}\} \\
&= \left(\sum_{n=1}^{\infty} \frac{x^n}{n} \right) \left(\sum_{n=1}^{\infty} \frac{x^n}{n} \right) \\
&\quad \left\{ \text{use the Cauchy product (2.65) where } a_n = \frac{1}{n} \text{ and } b_n = \frac{1}{n} \right\} \\
&= \sum_{n=1}^{\infty} x^{n+1} \left(\sum_{k=1}^{n} \frac{1}{k} \cdot \frac{1}{n-k+1} \right) \\
&\quad \left\{ \text{write } \frac{1}{k} \cdot \frac{1}{n-k+1} = \frac{1}{n+1} \left(\frac{1}{k} + \frac{1}{n-k+1} \right) \right\} \\
&= \sum_{n=1}^{\infty} \frac{x^{n+1}}{n+1} \left(\sum_{k=1}^{n} \frac{1}{k} + \sum_{k=1}^{n} \frac{1}{n-k+1} \right)
\end{aligned}$$

$$\left\{\text{use} \sum_{k=1}^{n} \frac{1}{n-k+1} = \sum_{k=1}^{n} \frac{1}{k} \text{ given in (1.3)}\right\}$$

$$= \sum_{n=1}^{\infty} \frac{x^{n+1}}{n+1} \left(2 \sum_{k=1}^{n} \frac{1}{k}\right)$$

$$= \sum_{n=1}^{\infty} \frac{x^{n+1}}{n+1} (2H_n)$$

$$\{\text{let the index } n \text{ start from } 0\text{, since } H_0 = 0\}$$

$$= 2 \sum_{n=0}^{\infty} \frac{H_n}{n+1} x^{n+1}$$

$$\{\text{shift the index } n \text{ by } -1\}$$

$$= 2 \sum_{n=1}^{\infty} \frac{H_{n-1}}{n} x^n \tag{2.7}$$

$$= 2 \sum_{n=1}^{\infty} \frac{H_n - \frac{1}{n}}{n} x^n$$

$$= 2 \sum_{n=1}^{\infty} \frac{H_n}{n} x^n - 2 \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

$$= 2 \sum_{n=1}^{\infty} \frac{H_n}{n} x^n - 2 \operatorname{Li}_2(x).$$

2.1.3 $\sum_{n=1}^{\infty} \frac{H_n}{n^2} x^n$

Let $|x| \le 1$. Then the following identity holds:

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} x^n = \operatorname{Li}_3(x) - \operatorname{Li}_3(1-x) + \ln(1-x) \operatorname{Li}_2(1-x)$$
$$+ \frac{1}{2} \ln(x) \ln^2(1-x) + \zeta(3). \tag{2.8}$$

Proof.

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} x^n = \sum_{n=1}^{\infty} \frac{H_n}{n} \frac{x^n}{n} = \sum_{n=1}^{\infty} \frac{H_n}{n} \left(\int x^{n-1} \mathrm{d}x\right)$$

$$= \int \frac{1}{x} \left(\sum_{n=1}^{\infty} \frac{H_n}{n} x^n\right) \mathrm{d}x$$

$$\begin{aligned}
&\{\text{recall the generating function (2.6)}\}\\
&= \int \frac{1}{x}\left(\mathrm{Li}_2(x) + \frac{1}{2}\ln^2(1-x)\right)\mathrm{d}x\\
&= \int \frac{\mathrm{Li}_2(x)}{x}\mathrm{d}x + \frac{1}{2}\int \frac{\ln^2(1-x)}{x}\mathrm{d}x.
\end{aligned}$$

The first integral is $\mathrm{Li}_3(x)$. For the second, force integration by parts twice,

$$\begin{aligned}
\int \frac{\ln^2(1-x)}{x}\mathrm{d}x &= \ln(x)\ln^2(1-x) + 2\int \frac{\ln(x)\ln(1-x)}{1-x}\mathrm{d}x\\
&= \ln(x)\ln^2(1-x) + 2\left(\ln(1-x)\,\mathrm{Li}_2(1-x) + \int \frac{\mathrm{Li}_2(1-x)}{1-x}\mathrm{d}x\right)\\
&= \ln(x)\ln^2(1-x) + 2\ln(1-x)\,\mathrm{Li}_2(1-x) - 2\,\mathrm{Li}_3(1-x). \qquad (2.9)
\end{aligned}$$

Combining the two integrals, we get

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{H_n}{n^2}x^n &= \mathrm{Li}_3(x) - \mathrm{Li}_3(1-x) + \ln(1-x)\,\mathrm{Li}_2(1-x)\\
&\quad + \frac{1}{2}\ln(x)\ln^2(1-x) + c.
\end{aligned}$$

Set $x = 0$ to get $c = \zeta(3)$ and the proof os complete.

2.1.4 $\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n}x^n$

Let $-1 \le x < 1$. Then the following identity holds:

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n}x^n &= \mathrm{Li}_3(x) + 2\,\mathrm{Li}_3(1-x) - \ln(1-x)\,\mathrm{Li}_2(1-x)\\
&\quad - \zeta(2)\ln(1-x) - 2\zeta(3). \qquad (2.10)
\end{aligned}$$

Proof.

$$\begin{aligned}
\sum_{n=1}^{\infty}H_n^{(2)}\frac{x^n}{n} &= \sum_{n=1}^{\infty}H_n^{(2)}\left(\int x^{n-1}\mathrm{d}x\right)\\
&= \int \frac{1}{x}\left(\sum_{n=1}^{\infty}H_n^{(2)}x^n\right)\mathrm{d}x\\
&\{\text{set } a = 2 \text{ in (2.2) to get this sum}\}
\end{aligned}$$

$$
\begin{aligned}
&= \int \frac{1}{x}\left(\frac{\operatorname{Li}_2(x)}{1-x}\right) \mathrm{d}x \\
&= \int \frac{\operatorname{Li}_2(x)}{x} \mathrm{d}x + \underbrace{\int \frac{\operatorname{Li}_2(x)}{1-x} \mathrm{d}x}_{\text{IBP}} \\
&= \operatorname{Li}_3(x) - \ln(1-x)\operatorname{Li}_2(x) - \int \frac{\ln^2(1-x)}{x} \mathrm{d}x \\
&\qquad \{\text{this integral is evaluated in (2.9)}\} \\
&= \operatorname{Li}_3(x) + 2\operatorname{Li}_3(1-x) - \ln(1-x)[\operatorname{Li}_2(x) + \ln(x)\ln(1-x)] \\
&\qquad -2\ln(1-x)\operatorname{Li}_2(1-x) + c \\
&\qquad \{\text{make use of the reflection formula (1.84) for the third term}\} \\
&= \operatorname{Li}_3(x) + 2\operatorname{Li}_3(1-x) - \ln(1-x)\operatorname{Li}_2(1-x) - \zeta(2)\ln(1-x) + c,
\end{aligned}
$$

where $c = -2\zeta(3)$ by putting $x = 0$.
For a different method, substitute (2.8) in (2.68).

2.1.5 $\sum_{n=1}^{\infty}(H_n^2 - H_n^{(2)})x^n$

Let $|x| < 1$. Then the following identity holds:

$$
\sum_{n=1}^{\infty}(H_n^2 - H_n^{(2)})x^n = \frac{\ln^2(1-x)}{1-x}. \tag{2.11}
$$

Proof. Set $a_n = H_n^2 - H_n^{(2)}$ in (2.1),

$$
\begin{aligned}
\sum_{n=1}^{\infty}(H_n^2 - H_n^{(2)})x^n &= \frac{1}{1-x}\sum_{n=1}^{\infty}\left((H_n^2 - H_n^{(2)}) - (H_{n-1}^2 - H_{n-1}^{(2)})\right)x^n \\
&\qquad \left\{\text{write } H_{n-1}^2 = \left(H_n - \frac{1}{n}\right)^2 \text{ and } H_{n-1}^{(2)} = H_n^{(2)} - \frac{1}{n^2}\right\} \\
&= \frac{2}{1-x}\sum_{n=1}^{\infty}\left(H_n - \frac{1}{n}\right)\frac{x^n}{n} \\
&= \frac{2}{1-x}\sum_{n=1}^{\infty}\frac{H_{n-1}}{n}x^n \\
&\qquad \{\text{substitute the result (2.7)}\} \\
&= \frac{2}{1-x}\left(\frac{1}{2}\ln^2(1-x)\right)
\end{aligned}
$$

$$= \frac{\ln^2(1-x)}{1-x}.$$

This result is also proved in (2.60).

Furthermore, let the index in (2.11) start from 0, since $H_0^2 - H_0^{(2)} = 0$,

$$\frac{\ln^2(1-x)}{1-x} = \sum_{n=0}^{\infty}(H_n^2 - H_n^{(2)})x^n$$

$$\{\text{shift the index by } -1\}$$

$$= \sum_{n=1}^{\infty}(H_{n-1}^2 - H_{n-1}^{(2)})x^{n-1}$$

$$= \sum_{n=1}^{\infty}\left(\left(H_n - \frac{1}{n}\right)^2 - \left(H_n^{(2)} - \frac{1}{n^2}\right)\right)x^{n-1}$$

$$= \sum_{n=1}^{\infty}\left(H_n^2 - H_n^{(2)} - \frac{2H_n}{n} + \frac{2}{n^2}\right)x^{n-1}. \tag{2.12}$$

Check [32, p. 355] for an alternative proof.

2.1.6 $\sum_{n=1}^{\infty} \frac{H_n^2 - H_n^{(2)}}{n} x^n$

Let $-1 \le x < 1$. Then the following identity holds:

$$\sum_{n=1}^{\infty}(H_n^2 - H_n^{(2)})\frac{x^n}{n} = 2\ln(1-x)\operatorname{Li}_2(1-x) - 2\operatorname{Li}_3(1-x)$$
$$+ \ln(x)\ln^2(1-x) - \frac{1}{3}\ln^3(1-x) + 2\zeta(3). \tag{2.13}$$

Proof.

$$\sum_{n=1}^{\infty}(H_n^2 - H_n^{(2)})\frac{x^n}{n} = \sum_{n=1}^{\infty}(H_n^2 - H_n^{(2)})\left(\int x^{n-1}\mathrm{d}x\right)$$

$$= \int \frac{1}{x}\left(\sum_{n=1}^{\infty}(H_n^2 - H_n^{(2)})x^n\right)\mathrm{d}x$$

$$\{\text{recall the generating function (2.11)}\}$$

$$= \int \frac{1}{x}\left(\frac{\ln^2(1-x)}{1-x}\right)\mathrm{d}x$$

$$= \int \frac{\ln^2(1-x)}{x}\mathrm{d}x + \int \frac{\ln^2(1-x)}{1-x}\mathrm{d}x$$

$$\{\text{the first integral is found in (2.9)}\}$$

$$= \ln(x)\ln^2(1-x) + 2\ln(1-x)\operatorname{Li}_2(1-x) - 2\operatorname{Li}_3(1-x)$$
$$-\frac{1}{3}\ln^3(1-x) + c.$$

On setting $x = 0$, we get $c = 2\zeta(3)$ and the proof is finalized.

2.1.7 $\sum_{n=1}^{\infty} \frac{H_n^2}{n} x^n$

Let $-1 \le x < 1$. Then the following identity holds:

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n} x^n = \operatorname{Li}_3(x) - \ln(1-x)\operatorname{Li}_2(x) - \frac{1}{3}\ln^3(1-x). \tag{2.14}$$

Proof. The proof follows on combining the results (2.10) and (2.13).
Check [32, p. 349–350] for another approach.

2.1.8 $\sum_{n=1}^{\infty} \frac{H_n}{n^3} x^n$

Let $-1 \le x < 1$. Then the following identity holds:

$$\sum_{n=1}^{\infty} \frac{H_n}{n^3} x^n = \operatorname{Li}_4\left(\frac{x}{x-1}\right) - \operatorname{Li}_4(1-x) + 2\operatorname{Li}_4(x) - \ln(1-x)\operatorname{Li}_3(1-x)$$
$$+\zeta(3)\ln(1-x) + \frac{1}{2}\zeta(2)\ln^2(1-x) - \frac{1}{6}\ln(x)\ln^3(1-x) + \frac{1}{24}\ln^4(1-x) + \zeta(4). \tag{2.15}$$

Proof. Replace z with $1-x$ in (1.90),

$$\operatorname{Li}_3(1-x) + \operatorname{Li}_3(x) + \operatorname{Li}_3\left(\frac{x}{x-1}\right)$$
$$= \zeta(3) + \frac{1}{6}\ln^3(1-x) + \zeta(2)\ln(1-x) - \frac{1}{2}\ln^2(1-x)\ln(x).$$

Divide both sides by $x(1-x)$ then integrate,

$$\begin{aligned}
&\int \frac{\operatorname{Li}_3(1-x)}{x(1-x)}\mathrm{d}x + \int \frac{\operatorname{Li}_3(x)}{x(1-x)}\mathrm{d}x + \int \frac{\operatorname{Li}_3\left(\frac{x}{x-1}\right)}{x(1-x)}\mathrm{d}x \\
&= \zeta(3)\int \frac{\mathrm{d}x}{x(1-x)} + \frac{1}{6}\int \frac{\ln^3(1-x)}{x(1-x)}\mathrm{d}x + \zeta(2)\int \frac{\ln(1-x)}{x(1-x)}\mathrm{d}x \\
&\quad -\frac{1}{2}\int \frac{\ln^2(1-x)\ln(x)}{x(1-x)}\mathrm{d}x.
\end{aligned}$$

First integral:

$$\begin{aligned}
\int \frac{\operatorname{Li}_3(1-x)}{x(1-x)}\mathrm{d}x &= \underbrace{\int \frac{\operatorname{Li}_3(1-x)}{x}\mathrm{d}x}_{\text{IBP}} + \int \frac{\operatorname{Li}_3(1-x)}{1-x}\mathrm{d}x \\
&= \ln(x)\operatorname{Li}_3(1-x) + \int \frac{\ln(x)\operatorname{Li}_2(1-x)}{1-x}\mathrm{d}x - \operatorname{Li}_4(1-x) \\
&= \ln(x)\operatorname{Li}_3(1-x) + \frac{1}{2}\operatorname{Li}_2^2(1-x) - \operatorname{Li}_4(1-x).
\end{aligned}$$

Second integral:

$$\begin{aligned}
\int \frac{\operatorname{Li}_3(x)}{x(1-x)}\mathrm{d}x &= \int \frac{\operatorname{Li}_3(x)}{x}\mathrm{d}x + \underbrace{\int \frac{\operatorname{Li}_3(x)}{1-x}\mathrm{d}x}_{\text{IBP}} \\
&= \operatorname{Li}_4(x) - \ln(1-x)\operatorname{Li}_3(x) + \int \frac{\ln(1-x)\operatorname{Li}_2(x)}{x}\mathrm{d}x \\
&= \operatorname{Li}_4(x) - \ln(1-x)\operatorname{Li}_3(x) - \frac{1}{2}\operatorname{Li}_2^2(x).
\end{aligned}$$

Third integral: Let $x = \frac{y}{y-1}$,

$$\int \frac{\operatorname{Li}_3\left(\frac{x}{x-1}\right)}{x(1-x)}\mathrm{d}x = \int \frac{\operatorname{Li}_3(y)}{y}\mathrm{d}y = \operatorname{Li}_4(y) = \operatorname{Li}_4\left(\frac{x}{x-1}\right).$$

Fourth integral:

$$\int \frac{\mathrm{d}x}{x(1-x)} = \int \frac{\mathrm{d}x}{x} + \int \frac{\mathrm{d}x}{1-x} = \ln(x) - \ln(1-x).$$

Fifth integral:

$$\int \frac{\ln^3(1-x)}{x(1-x)}\mathrm{d}x = \int \frac{\ln^3(1-x)}{x}\mathrm{d}x + \int \frac{\ln^3(1-x)}{1-x}\mathrm{d}x$$

$$= \int \frac{\ln^3(1-x)}{x}\mathrm{d}x - \frac{1}{4}\ln^4(1-x).$$

Sixth integral: Let $x = \frac{y}{y-1}$,

$$\int \frac{\ln(1-x)}{x(1-x)}\mathrm{d}x = -\int \frac{\ln(1-y)}{y}\mathrm{d}y = \mathrm{Li}_2(y) = \mathrm{Li}_2\left(\frac{x}{x-1}\right).$$

Seventh integral:

$$\int \frac{\ln^2(1-x)\ln(x)}{x(1-x)}\mathrm{d}x = \underbrace{\int \frac{\ln^2(1-x)\ln(x)}{1-x}\mathrm{d}x}_{\text{IBP}} + \int \frac{\ln^2(1-x)\ln(x)}{x}\mathrm{d}x$$

$$= -\frac{1}{3}\ln^3(1-x)\ln(x) + \frac{1}{3}\int \frac{\ln^3(1-x)}{x}\mathrm{d}x + \int \frac{\ln^2(1-x)\ln(x)}{x}\mathrm{d}x.$$

By expanding $\ln^2(1-x)$ in series given in (2.7), we have

$$\int \frac{\ln^2(1-x)\ln(x)}{x}\mathrm{d}x = 2\sum_{n=1}^{\infty}\frac{H_{n-1}}{n}\int x^{n-1}\ln(x)\mathrm{d}x$$

$$\overset{\text{IBP}}{=} 2\sum_{n=1}^{\infty}\frac{H_n - \frac{1}{n}}{n}\left(\frac{\ln(x)}{n}x^n - \frac{x^n}{n^2}\right)$$

$$= 2\ln(x)\sum_{n=1}^{\infty}\frac{H_n}{n^2}x^n - 2\sum_{n=1}^{\infty}\frac{H_n}{n^3}x^n - 2\ln(x)\,\mathrm{Li}_3(x) + 2\,\mathrm{Li}_4(x)$$

$$\{\text{substitute the result (2.8)}\}$$

$$= 2\,\mathrm{Li}_4(x) - 2\ln(x)\,\mathrm{Li}_3(1-x) + 2\ln(x)\ln(1-x)\,\mathrm{Li}_2(1-x)$$

$$+\ln^2(x)\ln^2(1-x) + 2\zeta(3)\ln(x) - 2\sum_{n=1}^{\infty}\frac{H_n}{n^3}x^n.$$

Plug this integral back,

$$\int \frac{\ln^2(1-x)\ln(x)}{x(1-x)}\mathrm{d}x$$

$$= 2\,\mathrm{Li}_4(x) - \frac{1}{3}\ln^3(1-x)\ln(x) - 2\ln(x)\,\mathrm{Li}_3(1-x) + \ln^2(x)\ln^2(1-x)$$

$$+2\ln(x)\ln(1-x)\,\mathrm{Li}_2(1-x) + 2\zeta(3)\ln(x)$$

$$-2\sum_{n=1}^{\infty}\frac{H_n}{n^3}x^n + \frac{1}{3}\int \frac{\ln^3(1-x)}{x}\mathrm{d}x.$$

By collecting all seven integrals, the integral $\int \frac{\ln^3(1-x)}{x}\mathrm{d}x$ nicely cancels out,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n}{n^3}x^n &= \operatorname{Li}_4\left(\frac{x}{x-1}\right) - \operatorname{Li}_4(1-x) + 2\operatorname{Li}_4(x) - \ln(1-x)\operatorname{Li}_3(1-x)\\ &+\frac{1}{24}\ln^4(1-x) - \frac{1}{6}\ln(x)\ln^3(1-x) + \zeta(3)\ln(1-x)\\ &+\frac{1}{2}\operatorname{Li}_2^2(1-x) + \ln(x)\ln(1-x)\operatorname{Li}_2(1-x) + \frac{1}{2}\ln^2(x)\ln^2(1-x)\\ &-\frac{1}{2}\operatorname{Li}_2^2(x) - \zeta(2)\operatorname{Li}_2\left(\frac{x}{x-1}\right).\end{aligned}$$

The last two lines can be further simplified:

$$\begin{aligned}&\frac{1}{2}\operatorname{Li}_2^2(1-x) + \ln(x)\ln(1-x)\operatorname{Li}_2(1-x) + \frac{1}{2}\ln^2(x)\ln^2(1-x)\\ &\quad -\frac{1}{2}\operatorname{Li}_2^2(x) - \zeta(2)\operatorname{Li}_2\left(\frac{x}{x-1}\right)\\ &= \frac{1}{2}\Big[\operatorname{Li}_2(1-x) + \ln(x)\ln(1-x)\Big]^2 - \frac{1}{2}\operatorname{Li}_2^2(x) - \zeta(2)\operatorname{Li}_2\left(\frac{x}{x-1}\right)\\ &\quad \{\text{use the dilogarithm reflection formula (1.80) for the first term}\}\\ &= \frac{1}{2}\Big[\zeta(2) - \operatorname{Li}_2(x)\Big]^2 - \frac{1}{2}\operatorname{Li}_2^2(x) - \zeta(2)\operatorname{Li}_2\left(\frac{x}{x-1}\right)\\ &= \frac{5}{4}\zeta(4) - \zeta(2)\operatorname{Li}_2(x) - \zeta(2)\operatorname{Li}_2\left(\frac{x}{x-1}\right)\\ &= \frac{5}{4}\zeta(4) - \zeta(2)\left[\operatorname{Li}_2(x) + \operatorname{Li}_2\left(\frac{x}{x-1}\right)\right]\\ &\quad \{\text{make use of Landen's dilogarithm identity (1.84)}\}\\ &= \frac{5}{4}\zeta(4) - \zeta(2)\Big[-\frac{1}{2}\ln^2(1-x)\Big]\\ &= \frac{5}{4}\zeta(4) + \frac{1}{2}\zeta(2)\ln^2(1-x).\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n}{n^3}x^n &= \operatorname{Li}_4\left(\frac{x}{x-1}\right) - \operatorname{Li}_4(1-x) + 2\operatorname{Li}_4(x) - \ln(1-x)\operatorname{Li}_3(1-x)\\ &+\frac{1}{24}\ln^4(1-x) - \frac{1}{6}\ln(x)\ln^3(1-x) + \zeta(3)\ln(1-x)\\ &+\frac{5}{4}\zeta(4) + \frac{1}{2}\zeta(2)\ln^2(1-x) + c.\end{aligned}$$

Set $x=0$, we get $c=-\frac{1}{4}\zeta(4)$ and this finalizes the proof.

2.1.9 $\sum_{n=1}^\infty \frac{H_n^{(2)}}{n^2}x^n$

Let $-1\le x<1$. Then the following identity holds:

$$\sum_{n=1}^\infty \frac{H_n^{(2)}}{n^2}x^n=-2\operatorname{Li}_4\left(\frac{x}{x-1}\right)+2\operatorname{Li}_4(1-x)-\operatorname{Li}_4(x)+\frac{1}{2}\operatorname{Li}_2^2(x)$$
$$+2\ln(1-x)\operatorname{Li}_3(1-x)-\frac{1}{12}\ln^4(1-x)-\zeta(2)\ln^2(1-x)$$
$$-2\zeta(3)\ln(1-x)+\frac{1}{3}\ln(x)\ln^3(1-x)-2\zeta(4). \tag{2.16}$$

Proof. Substitute (2.15) in (2.69).

2.1.10 $\sum_{n=1}^\infty \frac{H_n^{(3)}}{n}x^n$

Let $-1\le x<1$. Then the following identity holds:

$$\sum_{n=1}^\infty \frac{H_n^{(3)}}{n}x^n=\operatorname{Li}_4(x)-\ln(1-x)\operatorname{Li}_3(x)-\frac{1}{2}\operatorname{Li}_2^2(x). \tag{2.17}$$

Proof.

$$\sum_{n=1}^\infty H_n^{(3)}\frac{x^n}{n}=\sum_{n=1}^\infty H_n^{(3)}\left(\int x^{n-1}\mathrm{d}x\right)$$
$$=\int\frac{1}{x}\left(\sum_{n=1}^\infty H_n^{(3)}x^n\right)\mathrm{d}x$$
$$\{\text{set } a=3 \text{ in } (2.2) \text{ to get this sum}\}$$
$$=\int\frac{1}{x}\left(\frac{\operatorname{Li}_3(x)}{1-x}\right)\mathrm{d}x$$
$$=\int\frac{\operatorname{Li}_3(x)}{x}\mathrm{d}x+\underbrace{\int\frac{\operatorname{Li}_3(x)}{1-x}\mathrm{d}x}_{\text{IBP}}$$
$$=\operatorname{Li}_4(x)-\ln(1-x)\operatorname{Li}_3(x)+\int\frac{\ln(1-x)\operatorname{Li}_2(x)}{x}\mathrm{d}x$$

$$= \operatorname{Li}_4(x) - \ln(1-x)\operatorname{Li}_3(x) - \frac{1}{2}\operatorname{Li}_2^2(x) + c.$$

The proof completes on extracting $c = 0$ by setting $x = 0$.

2.1.11 $\sum_{n=1}^{\infty} H_n^3 x^n$

Let $|x| < 1$. Then the following identity holds:

$$\sum_{n=1}^{\infty} H_n^3 x^n = \frac{1}{1-x}\Bigg[\operatorname{Li}_3(x) + 3\operatorname{Li}_3(1-x) + \frac{3}{2}\ln(x)\ln^2(1-x) \\ -3\zeta(2)\ln(1-x) - \ln^3(1-x) - 3\zeta(3)\Bigg]. \tag{2.18}$$

Proof. Set $a_n = H_n^3$ in (2.1),

$$\begin{aligned}\sum_{n=1}^{\infty} H_n^3 x^n &= \frac{1}{1-x}\sum_{n=1}^{\infty}(H_n^3 - H_{n-1}^3)x^n \\ &= \frac{1}{1-x}\sum_{n=1}^{\infty}\left(H_n^3 - \left(H_n - \frac{1}{n}\right)^3\right)x^n \\ &= \frac{1}{1-x}\sum_{n=1}^{\infty}\left(\frac{3H_n^2}{n} - \frac{3H_n}{n^2} + \frac{1}{n^3}\right)x^n \\ &= \frac{1}{1-x}\left(3\sum_{n=1}^{\infty}\frac{H_n^2}{n}x^n - 3\sum_{n=1}^{\infty}\frac{H_n}{n^2}x^n + \sum_{n=1}^{\infty}\frac{x^n}{n^3}\right).\end{aligned}$$

Gathering the results (2.14) and (2.8) ends the proof.
A different method may be found in [32, p. 352–354].

2.1.12 $\sum_{n=1}^{\infty} \frac{H_n^2}{n^2} x^n$

Let $|x| \le 1$. Then the following identity holds:

$$\sum_{n=1}^{\infty}\frac{H_n^2}{n^2}x^n = \operatorname{Li}_4(x) - 2\operatorname{Li}_4(1-x) + 2\ln(1-x)\operatorname{Li}_3(1-x) + \frac{1}{2}\operatorname{Li}_2^2(x)$$

(continued)

$$-\ln^2(1-x)\operatorname{Li}_2(1-x)-\frac{1}{3}\ln(x)\ln^3(1-x)+2\zeta(4). \tag{2.19}$$

Proof.

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{H_n^2}{n^2}x^n=\sum_{n=1}^{\infty}\frac{H_n^2}{n}\frac{x^n}{n}&=\sum_{n=1}^{\infty}\frac{H_n^2}{n}\left(\int x^{n-1}\mathrm{d}x\right)\\
&=\int\frac{1}{x}\left(\sum_{n=1}^{\infty}\frac{H_n^2}{n}x^n\right)\mathrm{d}x\\
&\{\text{recall the generating function (2.14)}\}\\
&=\int\frac{1}{x}\left(\operatorname{Li}_3(x)-\ln(1-x)\operatorname{Li}_2(x)-\frac{1}{3}\ln^3(1-x)\right)\mathrm{d}x\\
&=\int\frac{\operatorname{Li}_3(x)}{x}\mathrm{d}x-\int\frac{\ln(1-x)\operatorname{Li}_2(x)}{x}\mathrm{d}x-\frac{1}{3}\int\frac{\ln^3(1-x)}{x}\mathrm{d}x\\
&=\operatorname{Li}_4(x)+\frac{1}{2}\operatorname{Li}_2^2(x)-\frac{1}{3}\int\frac{\ln^3(1-x)}{x}\mathrm{d}x
\end{aligned}$$

For the remaining integral, set $1-x=y$ then expand $\frac{1}{1-y}$ in series,

$$\begin{aligned}
\int\frac{\ln^3(1-x)}{x}\mathrm{d}x&=-\int\frac{\ln^3(y)}{1-y}\mathrm{d}y\\
&=-\sum_{n=1}^{\infty}\int y^{n-1}\ln^3(y)\mathrm{d}y\\
&\overset{\text{IBP}}{=}-\sum_{n=1}^{\infty}\left(\ln^3(y)\frac{y^n}{n}-3\ln^2(y)\frac{y^n}{n^2}+6\ln(y)\frac{y^n}{n^3}-6\frac{y^n}{n^4}\right)\\
&=\ln^3(y)\ln(1-y)+3\ln^2(y)\operatorname{Li}_2(y)-6\ln(y)\operatorname{Li}_3(y)+6\operatorname{Li}_4(y)\\
&\{\text{substitute } y=1-x \text{ back}\}\\
&=\ln^3(1-x)\ln(x)+3\ln^2(1-x)\operatorname{Li}_2(1-x)-6\ln(1-x)\operatorname{Li}_3(1-x)\\
&\quad+6\operatorname{Li}_4(1-x). \qquad (2.20)
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{H_n^2}{n^2}x^n&=\frac{1}{2}\operatorname{Li}_2^2(x)-\frac{1}{3}\ln^3(1-x)\ln(x)-\ln^2(1-x)\operatorname{Li}_2(1-x)\\
&+2\ln(1-x)\operatorname{Li}_3(1-x)-2\operatorname{Li}_4(1-x)+\operatorname{Li}_4(x)+c.
\end{aligned}$$

The proof follows on extracting $c=2\zeta(4)$ by setting $x=0$.

2.1.13 $\sum_{n=1}^{\infty} H_n H_n^{(2)} x^n$

Let $|x| < 1$. Then the following identity holds:

$$\sum_{n=1}^{\infty} H_n H_n^{(2)} x^n = \frac{1}{1-x}\left[\operatorname{Li}_3(x) + \operatorname{Li}_3(1-x) + \frac{1}{2}\ln(x)\ln^2(1-x) \right.$$
$$\left. -\zeta(2)\ln(1-x) - \zeta(3)\right]. \tag{2.21}$$

Proof. Set $a_n = H_n H_n^{(2)}$ in (2.1),

$$\begin{aligned}\sum_{n=1}^{\infty} H_n H_n^{(2)} x^n &= \frac{1}{1-x}\sum_{n=1}^{\infty}\left(H_n H_n^{(2)} - H_{n-1}H_{n-1}^{(2)}\right)x^n \\ &= \frac{1}{1-x}\sum_{n=1}^{\infty}\left(H_n H_n^{(2)} - \left(H_n - \frac{1}{n}\right)\left(H_n^{(2)} - \frac{1}{n^2}\right)\right)x^n \\ &= \frac{1}{1-x}\sum_{n=1}^{\infty}\frac{H_n}{n^2}x^n + \frac{1}{1-x}\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n}x^n - \frac{\operatorname{Li}_3(x)}{1-x}.\end{aligned}$$

Collect the results (2.8) and (2.10) to complete the proof.
Another approach may be found in [32, pp. 350–552].

2.1.14 $\sum_{n=1}^{\infty}(H_n^3 - 3H_n H_n^{(2)} + 2H_n^{(3)})x^n$

Let $|x| < 1$. Then the following identity holds:

$$\sum_{n=1}^{\infty}\left(H_n^3 - 3H_n H_n^{(2)} + 2H_n^{(3)}\right)x^n = -\frac{\ln^3(1-x)}{1-x}. \tag{2.22}$$

Proof. Set $a_n = H_n^3 - 3H_n H_n^{(2)} + 2H_n^{(3)}$ in (2.1),

$$\begin{aligned}&\sum_{n=1}^{\infty}\left(H_n^3 - 3H_n H_n^{(2)} + 2H_n^{(3)}\right)x^n \\ &= \frac{1}{1-x}\sum_{n=1}^{\infty}\left(H_n^3 - 3H_n H_n^{(2)} + 2H_n^{(3)} - H_{n-1}^3 + 3H_{n-1}H_{n-1}^{(2)} - 2H_{n-1}^{(3)}\right)x^n \\ &= \frac{3}{1-x}\sum_{n=1}^{\infty}\left(H_n^2 - H_n^{(2)}\right)\frac{x^n}{n} - \frac{6}{1-x}\sum_{n=1}^{\infty}\frac{H_n}{n^2}x^n + \frac{6\operatorname{Li}_3(x)}{1-x}.\end{aligned}$$

The proof ends on collecting the results (2.13) and (2.8).
For a different proof, check (2.60).

2.1.15 $\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n} x^n$

Let $-1 \le x < 1$. Then the following identity holds:

$$\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n} x^n = -\operatorname{Li}_4\left(\frac{x}{x-1}\right) + \operatorname{Li}_4(1-x) - \ln(1-x)\operatorname{Li}_3(1-x)$$
$$+\frac{1}{2}\ln^2(1-x)\operatorname{Li}_2(1-x) + \frac{1}{6}\ln(x)\ln^3(1-x) - \frac{1}{24}\ln^4(1-x) - \zeta(4). \tag{2.23}$$

Proof. By using the identity (2.62), we get

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{H_n^3 + 3H_n H_n^{(2)} + 2H_n^{(3)}}{n} y^n \\
&= \sum_{n=1}^{\infty} \left(-\int_0^1 x^{n-1}\ln^3(1-x)\mathrm{d}x\right) y^n \\
&= -\int_0^1 \frac{\ln^3(1-x)}{x}\left(\sum_{n=1}^{\infty}(xy)^n\right)\mathrm{d}x \\
&\quad \{\text{use the geometric series formula}\} \\
&= -\int_0^1 \frac{\ln^3(1-x)}{x}\left(\frac{xy}{1-xy}\right)\mathrm{d}x \\
&\overset{1-x=t}{=} -\int_0^1 \frac{y\ln^3(t)}{1-y+yt}\mathrm{d}t \\
&\quad \{\text{make use of the integral form (1.76)}\} \\
&= 6\operatorname{Li}_4\left(\frac{y}{y-1}\right).
\end{aligned} \tag{2.24}$$

To establish another relation, write $\frac{y^n}{n} = \int_0^y x^{n-1}\mathrm{d}x$ to get

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{H_n^3 - 3H_n H_n^{(2)} + 2H_n^{(3)}}{n} y^n \\
&= \sum_{n=1}^{\infty}\left(H_n^3 - 3H_n H_n^{(2)} + 2H_n^{(3)}\right)\left(\int_0^y x^{n-1}\mathrm{d}x\right)
\end{aligned}$$

$$= \int_0^y \frac{1}{x}\left(\sum_{n=1}^{\infty}\left(H_n^3 - 3H_nH_n^{(2)} + 2H_n^{(3)}\right)x^n\right)\mathrm{d}x$$

{recall the generating function (2.22)}

$$= \int_0^y \frac{1}{x}\left(-\frac{\ln^3(1-x)}{1-x}\right)\mathrm{d}x$$

$$= -\int_0^y \frac{\ln^3(1-x)}{x}\mathrm{d}x - \int_0^y \frac{\ln^3(1-x)}{1-x}\mathrm{d}x$$

{the first integral is given in in (2.20)}

$$\begin{aligned} &= \frac{1}{4}\ln^4(1-y) - \ln^3(1-y)\ln(y) - 3\ln^2(1-y)\operatorname{Li}_2(1-y) \\ &\quad +6\ln(1-y)\operatorname{Li}_3(1-y) - 6\operatorname{Li}_4(1-y) + 6\zeta(4). \end{aligned} \tag{2.25}$$

The proof finalizes on taking the difference of (2.24) and (2.25).

2.1.16 $\sum_{n=1}^{\infty}\frac{H_n^3}{n}x^n$

Let $-1 \le x < 1$. Then the following identity holds:

$$\begin{aligned} \sum_{n=1}^{\infty}\frac{H_n^3}{n}x^n &= -3\operatorname{Li}_4\left(\frac{x}{x-1}\right) - 3\operatorname{Li}_4(1-x) - 2\operatorname{Li}_4(x) + \operatorname{Li}_2^2(x) \\ &\quad +3\ln(1-x)\operatorname{Li}_3(1-x) + 2\ln(1-x)\operatorname{Li}_3(x) + \frac{1}{8}\ln^4(1-x) \\ &\quad -\frac{3}{2}\ln^2(1-x)\operatorname{Li}_2(1-x) - \frac{1}{2}\ln(x)\ln^3(1-x) + 3\zeta(4). \end{aligned} \tag{2.26}$$

Proof. Combine (2.24) and (2.25),

$$\begin{aligned} \sum_{n=1}^{\infty}\frac{H_n^3}{n}x^n &= -3\operatorname{Li}_4\left(\frac{x}{x-1}\right) - \frac{1}{2}\ln^3(1-y)\ln(x) - \frac{3}{2}\ln^2(1-x)\operatorname{Li}_2(1-x) \\ &\quad +3\ln(1-x)\operatorname{Li}_3(1-x) - 3\operatorname{Li}_4(1-x) + \frac{1}{8}\ln^4(1-x) \\ &\quad +3\zeta(4) - 2\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n}x^n. \end{aligned}$$

The last sum is evaluated in (2.17).

2.1.17 $\sum_{n=1}^{\infty}(H_n^4 - 6H_n^2H_n^{(2)} + 8H_nH_n^{(3)} + 3(H_n^{(2)})^2 - 6H_n^{(4)})x^n$

Let $|x| < 1$. Then the following identity holds:

$$\sum_{n=1}^{\infty}\left(H_n^4 - 6H_n^2H_n^{(2)} + 8H_nH_n^{(3)} + 3\left(H_n^{(2)}\right)^2 - 6H_n^{(4)}\right)x^n = \frac{\ln^4(1-x)}{1-x}. \tag{2.27}$$

Proof. Put $a_n = H_n^4 - 6H_n^2H_n^{(2)} + 8H_nH_n^{(3)} + 3\left(H_n^{(2)}\right)^2 - 6H_n^{(4)}$ in (2.1),

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left(H_n^4 - 6H_n^2H_n^{(2)} + 8H_nH_n^{(3)} + 3\left(H_n^{(2)}\right)^2 - 6H_n^{(4)}\right)x^n \\
&= \frac{1}{1-x}\sum_{n=1}^{\infty}\Big[\left(H_n^4 - H_{n-1}^4\right) - 6\left(H_n^2H_n^{(2)} - H_{n-1}^2H_{n-1}^{(2)}\right) \\
&+8\left(H_nH_n^{(3)} - H_{n-1}H_{n-1}^{(3)}\right) + 3\left(\left(H_n^{(2)}\right)^2 - \left(H_{n-1}^{(2)}\right)^2\right) \\
&-6\left(H_n^{(4)} - H_{n-1}^{(4)}\right)\Big]x^n \\
&= \frac{1}{1-x}\left[6\underbrace{\sum_{n=1}^{\infty}\left(\frac{4H_n}{n^3} + \frac{2H_n^{(2)}}{n^2} - \frac{6}{n^4}\right)x^n}_{S_1}\right. \\
&\left.+4\underbrace{\sum_{n=1}^{\infty}\left(\frac{H_n^3}{n} - \frac{3H_nH_n^{(2)}}{n} + \frac{2H_n^{(3)}}{n} - \frac{3H_n^2}{n^2} + \frac{3}{n^4}\right)x^n}_{S_2}\right].
\end{aligned}$$

The sum S_1 is the Cauchy product of $\operatorname{Li}_2^2(x)$ given in (2.69). For S_2, collect the results (2.14), (2.18), and (2.21) to get

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left(H_n^3 - 3H_nH_n^{(2)} + 2H_n^{(3)} - \frac{3H_n^2}{n} + \frac{3}{n^3}\right)x^n \\
&= -\frac{x\ln^3(1-x)}{1-x} + 3\ln(1-x)\operatorname{Li}_2(x).
\end{aligned}$$

Divide both sides of this identity by x then integrate using $\int x^{n-1}\mathrm{d}x = \frac{x^n}{n}$,

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{H_n^3}{n}-\frac{3H_nH_n^{(2)}}{n}+\frac{2H_n^{(3)}}{n}-\frac{3H_n^2}{n^2}+\frac{3}{n^4}\right)x^n &:= S_2\\ &= -\int\frac{\ln^3(1-x)}{1-x}\mathrm{d}x+3\int\frac{\ln(1-x)\operatorname{Li}_2(x)}{x}\mathrm{d}x\\ &= \frac{1}{4}\ln^4(1-x)-\frac{3}{2}\operatorname{Li}_2^2(x)+c.\end{aligned}$$

Setting $x=0$ gives $c=0$. Collect S_1 and S_2 to complete the proof. Different proofs can be found in (2.60) and [32, p. 355].

2.1.18 $\sum_{n=1}^{\infty}\overline{H}_n^{(a)}x^n$

Let $-1\le x<1$. Then the following identity holds:

$$\sum_{n=1}^{\infty}\overline{H}_n^{(a)}x^n = -\frac{\operatorname{Li}_a(-x)}{1-x}. \tag{2.28}$$

Proof (i). Since $\overline{H}_0^{(a)}=0$, it's valid to set $a_n=\overline{H}_n^{(a)}$ in (2.1),

$$\begin{aligned}\sum_{n=1}^{\infty}\overline{H}_n^{(a)}x^n &= \frac{1}{1-x}\sum_{n=1}^{\infty}\left(\overline{H}_n^{(a)}-\overline{H}_{n-1}^{(a)}\right)x^n\\ &\left\{\text{write } \overline{H}_n^{(a)}-\overline{H}_{n-1}^{(a)}=\frac{(-1)^n}{n^a}\text{ given in (1.134)}\right\}\\ &= \frac{-1}{1-x}\sum_{n=1}^{\infty}\frac{(-x)^n}{n^a}\\ &= -\frac{\operatorname{Li}_a(-x)}{1-x}.\end{aligned}$$

Proof (ii).

$$\begin{aligned}-\frac{\operatorname{Li}_a(-x)}{1-x} &= (-\operatorname{Li}_a(-x))\left(\frac{1}{1-x}\right)\\ &\{\text{expand both functions in Taylor series}\}\\ &= \left(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^a}x^n\right)\left(\sum_{n=1}^{\infty}x^{n-1}\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{x}\left(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^a}x^n\right)\left(\sum_{n=1}^{\infty}1x^n\right)\\
&\left\{\text{employ (2.65) where } a_n=\frac{(-1)^{n-1}}{n^a} \text{ and } b_n=1\right\}\\
&= \frac{1}{x}\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^a}\cdot 1\right)x^{n+1}\\
&= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^a}\right)x^{n}\\
&= \sum_{n=1}^{\infty}\overline{H}_n^{(a)}x^n.
\end{aligned}
$$

Setting $a=1$, we have

$$
\sum_{n=1}^{\infty}\overline{H}_n x^n = -\frac{\operatorname{Li}_1(-x)}{1-x}=\frac{\ln(1+x)}{1-x}, \quad |x|<1. \tag{2.29}
$$

2.1.19 $\sum_{n=1}^{\infty}\frac{\overline{H}_n}{n}x^n$

Let $-1 \le x < 1$. Then the following identity holds:

$$
\sum_{n=1}^{\infty}\frac{\overline{H}_n}{n}x^n = \operatorname{Li}_2\left(\frac{1-x}{2}\right)-\operatorname{Li}_2(-x)-\ln(2)\ln(1-x)-\operatorname{Li}_2\left(\frac{1}{2}\right). \tag{2.30}
$$

Proof. The following proof may be found in [8, p. 4]:

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{\overline{H}_n}{n}x^n &= \sum_{n=1}^{\infty}\overline{H}_n\left(\int x^{n-1}\mathrm{d}x\right)\\
&= \int\frac{1}{x}\left(\sum_{n=1}^{\infty}\overline{H}_n x^n\right)\mathrm{d}x\\
&\{\text{recall the generating function (2.29)}\}\\
&= \int\frac{1}{x}\left(\frac{\ln(1+x)}{1-x}\right)\mathrm{d}x\\
&= \int\frac{\ln(1+x)}{x}\mathrm{d}x+\int\frac{\ln(1+x)}{1-x}\mathrm{d}x
\end{aligned}
$$

$$\left\{\text{write } \ln(1+x) = \ln\left(\frac{1+x}{2}\right) + \ln(2) \text{ in the second integral}\right\}$$

$$= \int \frac{\ln(1+x)}{x}\mathrm{d}x + \int \frac{\ln\left(\frac{1+x}{2}\right)}{1-x}\mathrm{d}x + \int \frac{\ln(2)}{1-x}\mathrm{d}x$$

$$\left\{\text{note that } \frac{d}{dx}\operatorname{Li}_2\left(\frac{1-x}{2}\right) = \frac{\ln\left(\frac{1+x}{2}\right)}{1-x}\right\}$$

$$= -\operatorname{Li}_2(-x) + \operatorname{Li}_2\left(\frac{1-x}{2}\right) - \ln(2)\ln(1-x) + c.$$

The proof finishes on extracting $c = \operatorname{Li}_2\left(\frac{1}{2}\right)$ by setting $x = 0$.

2.1.20 $\sum_{n=1}^{\infty} \frac{\overline{H}_n}{n^2} x^n$

Let $-1 \le x < 1$. Then the following identity holds:

$$\sum_{n=1}^{\infty} \frac{\overline{H}_n}{n^2} x^n = \operatorname{Li}_3\left(\frac{2x}{1+x}\right) - \operatorname{Li}_3\left(\frac{x}{1+x}\right) - \operatorname{Li}_3\left(\frac{1+x}{2}\right) - \operatorname{Li}_3(-x)$$
$$- \operatorname{Li}_3(x) + \operatorname{Li}_3\left(\frac{1}{2}\right) + \ln(1+x)\left[\operatorname{Li}_2(x) + \operatorname{Li}_2\left(\frac{1}{2}\right) + \frac{1}{2}\ln(2)\ln(1+x)\right]. \tag{2.31}$$

Proof.

$$\sum_{n=1}^{\infty} \frac{\overline{H}_n}{n^2} x^n = \sum_{n=1}^{\infty} \frac{\overline{H}_n}{n}\left(\int x^{n-1}\mathrm{d}x\right)$$

$$= \int \frac{1}{x}\left(\sum_{n=1}^{\infty} \frac{\overline{H}_n}{n} x^n\right)\mathrm{d}x$$

$$\{\text{recall the generating function (2.30)}\}$$

$$= \int \frac{1}{x}\left(\operatorname{Li}_2\left(\frac{1-x}{2}\right) - \operatorname{Li}_2\left(\frac{1}{2}\right) - \operatorname{Li}_2(-x) - \ln(2)\ln(1-x)\right)\mathrm{d}x$$

$$= \underbrace{\int \frac{\operatorname{Li}_2\left(\frac{1-x}{2}\right) - \operatorname{Li}_2\left(\frac{1}{2}\right)}{x}\mathrm{d}x}_{\text{IBP}}$$

$$- \int \frac{\operatorname{Li}_2(-x)}{x}\mathrm{d}x - \ln(2)\int \frac{\ln(1-x)}{x}\mathrm{d}x$$

$$= \ln(x)\left(\operatorname{Li}_2\left(\frac{1-x}{2}\right) - \operatorname{Li}_2\left(\frac{1}{2}\right)\right) - \int \frac{\ln(x)\ln(1+x)}{1-x}\mathrm{d}x$$

$$\begin{aligned}
&+\ln(2)\int\frac{\ln(x)}{1-x}\mathrm{d}x-\int\frac{\operatorname{Li}_2(-x)}{x}\mathrm{d}x-\ln(2)\int\frac{\ln(1-x)}{x}\mathrm{d}x\\
&=\ln(x)\left(\operatorname{Li}_2\left(\frac{1-x}{2}\right)-\operatorname{Li}_2\left(\frac{1}{2}\right)\right)-\int\frac{\ln(x)\ln(1+x)}{1-x}\mathrm{d}x\\
&\quad+\ln(2)\operatorname{Li}_2(1-x)-\operatorname{Li}_3(-x)+\ln(2)\operatorname{Li}_2(x). \qquad (2.32)
\end{aligned}$$

For the remaining integral, set $a=\ln(x)$ and $b=\ln(1+x)$ in the algebraic identity:

$$2ab=a^2+b^2-(a-b)^2$$

to have

$$2\ln(x)\ln(1+x)=\ln^2(x)+\ln^2(1+x)-\ln^2\left(\frac{x}{1+x}\right).$$

Divide both sides by $1-x$ then integrate,

$$2\int\frac{\ln(x)\ln(1+x)}{1-x}\mathrm{d}x=\int\frac{\ln^2(x)}{1-x}\mathrm{d}x+\int\frac{\ln^2(1+x)}{1-x}\mathrm{d}x-\int\frac{\ln^2\left(\frac{x}{1+x}\right)}{1-x}\mathrm{d}x.$$

First integral: Expand $\frac{1}{1-x}$ in series,

$$\begin{aligned}
\int\frac{\ln^2(x)}{1-x}\mathrm{d}x&=\sum_{n=1}^{\infty}\int x^{n-1}\ln^2(x)\mathrm{d}x\\
&\overset{\text{IBP}}{=}\sum_{n=1}^{\infty}\left(\ln^2(x)\frac{x^n}{n}-2\ln(x)\frac{x^n}{n^2}+2\frac{x^n}{n^3}\right)\\
&=-\ln^2(x)\ln(1-x)-2\ln(x)\operatorname{Li}_2(x)+2\operatorname{Li}_3(x). \qquad (2.33)
\end{aligned}$$

Second integral: Substitute $1+x=y$,

$$\begin{aligned}
\int\frac{\ln^2(1+x)}{1-x}\mathrm{d}x&=\int\frac{\ln^2(y)}{2-y}\mathrm{d}y\\
&\left\{\text{expand }\frac{1}{2-y}\text{ in Taylor series as }\sum_{n=1}^{\infty}\frac{y^{n-1}}{2^n}\right\}\\
&=\sum_{n=1}^{\infty}\frac{1}{2^n}\int y^{n-1}\ln^2(y)\mathrm{d}y\\
&=\sum_{n=1}^{\infty}\frac{1}{2^n}\left(\ln^2(y)\frac{y^n}{n}-2\ln(y)\frac{y^n}{n^2}+2\frac{y^n}{n^3}\right)\\
&=-\ln^2(y)\ln\left(1-\frac{y}{2}\right)-2\ln(y)\operatorname{Li}_2\left(\frac{y}{2}\right)+2\operatorname{Li}_3\left(\frac{y}{2}\right)\\
&\{\text{substitute }y=1+x\text{ back}\}
\end{aligned}$$

$$= -\ln^2(1+x)\ln\left(\frac{1-x}{2}\right) - 2\ln(1+x)\operatorname{Li}_2\left(\frac{1+x}{2}\right) + 2\operatorname{Li}_3\left(\frac{1+x}{2}\right).$$

Third integral: Substitute $\frac{x}{1+x} = t$,

$$\begin{aligned}\int \frac{\ln^2\left(\frac{x}{1+x}\right)}{1-x}\mathrm{d}x &= \int \frac{\ln^2(t)}{(1-t)(1-2t)}\mathrm{d}t\\ &= 2\int \frac{\ln^2(t)}{1-2t}\mathrm{d}t - \int \frac{\ln^2(t)}{1-t}\mathrm{d}t\\ &= -\ln^2(t)\ln(1-2t) - 2\ln(t)\operatorname{Li}_2(2t) + 2\operatorname{Li}_3(2t)\\ &\quad + \ln^2(t)\ln(1-t) + 2\ln(t)\operatorname{Li}_2(t) - 2\operatorname{Li}_3(t)\\ &= -\ln^2(t)\ln\left(\frac{1-2t}{1-t}\right) - 2\ln(t)\left(\operatorname{Li}_2(2t) - \operatorname{Li}_2(t)\right) + 2\operatorname{Li}_3(2t) - 2\operatorname{Li}_3(t)\\ &\quad \left\{\text{substitute } t = \frac{x}{1+x} \text{ back}\right\}\\ &= 2\operatorname{Li}_3\left(\frac{2x}{1+x}\right) - 2\operatorname{Li}_3\left(\frac{x}{1+x}\right) - \ln^2\left(\frac{x}{1+x}\right)\ln(1-x)\\ &\quad -2\ln\left(\frac{x}{1+x}\right)\left(\operatorname{Li}_2\left(\frac{2x}{1+x}\right) - \operatorname{Li}_2\left(\frac{x}{1+x}\right)\right).\end{aligned}$$

Gather the three integrals then divide by 2,

$$\begin{aligned}\int \frac{\ln(x)\ln(1+x)}{1-x}\mathrm{d}x &= \operatorname{Li}_3\left(\frac{x}{1+x}\right) - \operatorname{Li}_3\left(\frac{2x}{1+x}\right) + \operatorname{Li}_3\left(\frac{1+x}{2}\right)\\ &\quad + \operatorname{Li}_3(x) + \ln\left(\frac{x}{1+x}\right)\left(\operatorname{Li}_2\left(\frac{2x}{1+x}\right) - \operatorname{Li}_2\left(\frac{x}{1+x}\right)\right)\\ &\quad - \ln(1+x)\operatorname{Li}_2\left(\frac{1+x}{2}\right) - \ln(x)\operatorname{Li}_2(x) - \frac{1}{2}\ln^2(x)\ln(1-x)\\ &\quad -\frac{1}{2}\ln^2(1+x)\ln\left(\frac{1-x}{2}\right) + \frac{1}{2}\ln^2\left(\frac{x}{1+x}\right)\ln(1-x).\end{aligned}$$

Substitute this integral in (2.32) then factor $\ln(1+x)$, $\ln(x)$, and $\ln(2)$ out,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\overline{H}_n}{n^2}x^n &= \operatorname{Li}_3\left(\frac{2x}{1+x}\right) - \operatorname{Li}_3\left(\frac{x}{1+x}\right) - \operatorname{Li}_3\left(\frac{1+x}{2}\right) - \operatorname{Li}_3(-x)\\ &\quad - \operatorname{Li}_3(x) + \ln(1+x)\left[\operatorname{Li}_2\left(\frac{2x}{1+x}\right) - \operatorname{Li}_2\left(\frac{x}{1+x}\right) + \operatorname{Li}_2\left(\frac{1+x}{2}\right)\right]\\ &\quad - \ln(x)\left[\operatorname{Li}_2\left(\frac{2x}{1+x}\right) - \operatorname{Li}_2\left(\frac{x}{1+x}\right) - \operatorname{Li}_2\left(\frac{1-x}{2}\right) - \operatorname{Li}_2(x)\right.\end{aligned}$$

$$+\operatorname{Li}_2\left(\frac{1}{2}\right)\Bigg] + \ln(2)[\operatorname{Li}_2(x)+\operatorname{Li}_2(1-x)] + \frac{1}{2}\ln^2(1+x)\ln\left(\frac{1-x}{2}\right)$$
$$-\frac{1}{2}\ln^2\left(\frac{x}{1+x}\right)\ln(1-x)+\frac{1}{2}\ln^2(x)\ln(1-x)+c$$
$$\{\text{substitute the relations (1.88), (1.89), and (1.80)}\}$$
$$=\operatorname{Li}_3\left(\frac{2x}{1+x}\right)-\operatorname{Li}_3\left(\frac{x}{1+x}\right)-\operatorname{Li}_3\left(\frac{1+x}{2}\right)-\operatorname{Li}_3(-x)$$
$$-\operatorname{Li}_3(x)+\ln(1+x)\left[\operatorname{Li}_2(x)+\operatorname{Li}_2\left(\frac{1}{2}\right)+\ln(2)\ln(1+x)\right]$$
$$-\ln(x)\left[\ln(1-x)\ln(1+x)-\ln(2)\ln(1-x)\right]$$
$$+\ln(2)[\zeta(2)-\ln(x)\ln(1-x)]+\frac{1}{2}\ln^2(1+x)\ln\left(\frac{1-x}{2}\right)$$
$$-\frac{1}{2}\ln^2\left(\frac{x}{1+x}\right)\ln(1-x)+\frac{1}{2}\ln^2(x)\ln(1-x)+c,$$

where $c=\operatorname{Li}_3\left(\frac{1}{2}\right)-\ln(2)\zeta(2)$ by setting $x=0$.
The proof completes on simplifying the last three lines to $-\frac{1}{2}\ln(2)\ln^2(1+x)$.
A different proof may be found in [8, p. 9].

2.1.21 $\sum_{n=1}^{\infty} H_{\frac{n}{2}} x^n$

Let $|x|<1$. Then the following identity holds:

$$\sum_{n=1}^{\infty} H_{\frac{n}{2}} x^n = \frac{-2\ln(2)x}{1-x^2}-\frac{2\ln(1-x)}{1-x^2}. \tag{2.34}$$

The following proof is due to Wolfgang Hintze and it may be found in [12]:
Proof. start with considering the integral form of the harmonic number,

$$\sum_{n=1}^{\infty} H_{\frac{n}{2}} x^n = \sum_{n=1}^{\infty}\left(\int_0^1 \frac{1-y^{\frac{n}{2}}}{1-y}\mathrm{d}y\right)x^n$$
$$=\int_0^1 \frac{1}{1-y}\left(\sum_{n=1}^{\infty} x^n - \sum_{n=1}^{\infty}(x\sqrt{y})^n\right)\mathrm{d}y$$
$$=\int_0^1 \frac{1}{1-y}\left(\frac{x}{1-x}-\frac{x\sqrt{y}}{1-x\sqrt{y}}\right)\mathrm{d}y$$
$$\overset{\sqrt{y}=u}{=}\int_0^1 \frac{2u}{1-u^2}\left(\frac{x}{1-x}-\frac{xu}{1-xu}\right)\mathrm{d}u$$

$$
\begin{aligned}
&= \int_0^1 \frac{2u}{1-u^2}\left(\frac{x(1-u)}{(1-x)(1-xu)}\right)\mathrm{d}u\\
&\int_0^1 \frac{2xu}{(1-x)(1+u)(1-xu)}\mathrm{d}u\\
&= \frac{-2x}{1-x^2}\int_0^1 \frac{\mathrm{d}u}{1+u} - \frac{2}{1-x^2}\int_0^1 \frac{-x}{1-xu}\mathrm{d}u\\
&= \frac{-2x}{1-x^2}\ln(1+u)\bigg|_0^1 - \frac{2}{1-x^2}\ln(1-xu)\bigg|_0^1\\
&= -\frac{2x\ln(2)}{1-x^2} - \frac{2\ln(1-x)}{1-x^2}.
\end{aligned}
$$

2.1.22 $\sum_{n=1}^{\infty} \frac{H_{\frac{n}{2}}}{n} x^n$

Let $-1 \leq x < 1$. Then the following identity holds:

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{H_{\frac{n}{2}}}{n} x^n = \operatorname{Li}_2\left(\frac{1-x}{2}\right) - \operatorname{Li}_2\left(\frac{1}{2}\right) + 2\operatorname{Li}_2(x) + \frac{1}{2}\ln^2(1-x)\\
+ \ln(1+x)\ln\left(\frac{1-x}{2}\right). \qquad (2.35)
\end{aligned}
$$

The following proof is also due to Wolfgang Hintze:

Proof.

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{H_{\frac{n}{2}}}{n} x^n &= \sum_{n=1}^{\infty} H_{\frac{n}{2}} \left(\int x^{n-1}\mathrm{d}x\right)\\
&= \int \frac{1}{x}\left(\sum_{n=1}^{\infty} H_{\frac{n}{2}} x^n\right)\mathrm{d}x\\
&\quad \{\text{recall the generating function (2.34)}\}\\
&= \int \frac{1}{x}\left(\frac{-2\ln(2)x}{1-x^2} - \frac{2\ln(1-x)}{1-x^2}\right)\mathrm{d}x\\
&= -2\ln(2)\int \frac{\mathrm{d}x}{1-x^2} - \int \frac{2\ln(1-x)}{x(1-x^2)}\mathrm{d}x\\
&= \ln(2)\int -\frac{2\,\mathrm{d}x}{1-x^2} - 2\int \frac{\ln(1-x)}{x}\mathrm{d}x + \int \frac{\ln(1-x)}{1+x}\mathrm{d}x - \int \frac{\ln(1-x)}{1-x}\mathrm{d}x\\
&= \ln(2)\ln\left(\frac{1-x}{1+x}\right) + 2\operatorname{Li}_2(x) + \int \frac{\ln(1-x)}{1+x}\mathrm{d}x + \frac{1}{2}\ln^2(1-x).
\end{aligned}
$$

For the remaining integral, apply integration by parts,

$$\begin{aligned}\int \frac{\ln(1-x)}{1+x}\mathrm{d}x &= \ln(1+x)\ln(1-x) + \int \frac{\ln(1+x)}{1-x}\mathrm{d}x \\ &\left\{\text{write } \ln(1+x) = \ln\left(\frac{1+x}{2}\right) + \ln(2)\right\} \\ &= \ln(1+x)\ln(1-x) + \int \frac{\ln\left(\frac{1+x}{2}\right)}{1-x}\mathrm{d}x + \int \frac{\ln(2)}{1-x}\mathrm{d}x \\ &= \ln(1+x)\ln(1-x) + \operatorname{Li}_2\left(\frac{1-x}{2}\right) - \ln(2)\ln(1-x) \\ &= \operatorname{Li}_2\left(\frac{1-x}{2}\right) + \ln(1+x)\ln\left(\frac{1-x}{2}\right).\end{aligned}$$

Plug this integral back in,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{H_{\frac{n}{2}}}{n}x^n = \operatorname{Li}_2\left(\frac{1-x}{2}\right) + 2\operatorname{Li}_2(x) + \frac{1}{2}\ln^2(1-x) \\ + \ln(1+x)\ln\left(\frac{1-x}{2}\right) + c.\end{aligned}$$

The proof follows on extracting $c = -\operatorname{Li}_2(\frac{1}{2})$ by setting $x = 0$.

2.1.23 $\sum_{n=1}^{\infty} \frac{H_{\frac{n}{2}}}{n^2}x^n$

Let $-1 \le x < 1$. Then the following identity holds:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{H_{\frac{n}{2}}}{n^2}x^n &= \operatorname{Li}_3\left(\frac{2x}{1+x}\right) - \operatorname{Li}_3\left(\frac{x}{1+x}\right) - \operatorname{Li}_3\left(\frac{1+x}{2}\right) + \operatorname{Li}_3\left(\frac{1}{2}\right) \\ &+ \ln(1+x)\left[\operatorname{Li}_2\left(\frac{1}{1+x}\right) + \operatorname{Li}_2(x) + \operatorname{Li}_2\left(\frac{1}{2}\right) - \frac{1}{2}\ln(1+x)\ln\left(\frac{x}{2}\right)\right] \\ &- \frac{1}{2}\operatorname{Li}_3(1-x^2) + \frac{1}{2}\ln(1-x^2)\operatorname{Li}_2(1-x^2) + \ln(2)[\operatorname{Li}_2(-x) - \operatorname{Li}_2(x)] \\ &+ \operatorname{Li}_3(x) + \frac{1}{3}\ln^3(1+x) + \frac{1}{2}\ln(x)\ln^2(1-x^2) - \frac{1}{2}\zeta(3) + \operatorname{Li}_3\left(\frac{1}{1+x}\right).\end{aligned} \tag{2.36}$$

Proof. Bring back (2.35):

$$\sum_{n=1}^{\infty}\frac{H_{\frac{n}{2}}}{n}x^n=\operatorname{Li}_2\left(\frac{1-x}{2}\right)-\operatorname{Li}_2\left(\frac{1}{2}\right)+2\operatorname{Li}_2(x)-\ln(2)\ln(1+x)$$
$$+\frac{1}{2}\ln^2(1-x)+\ln(1+x)\ln(1-x)$$
$$\left\{\text{use } \frac{1}{2}\ln^2(1-x)+\ln(1+x)\ln(1-x)=\frac{1}{2}\ln^2(1-x^2)-\frac{1}{2}\ln^2(1+x)\right\}$$
$$=\operatorname{Li}_2\left(\frac{1-x}{2}\right)-\operatorname{Li}_2\left(\frac{1}{2}\right)+2\operatorname{Li}_2(x)-\ln(2)\ln(1+x)$$
$$+\frac{1}{2}\ln^2(1-x^2)-\frac{1}{2}\ln^2(1+x).$$

Take the difference of this identity and (2.30) then replace x with y, we reach

$$\sum_{n=1}^{\infty}\frac{H_{\frac{n}{2}}}{n}y^n-\sum_{n=1}^{\infty}\frac{\overline{H}_n}{n}y^n=2\operatorname{Li}_2(y)+\operatorname{Li}_2(-y)+\frac{1}{2}\ln^2(1-y^2)$$
$$-\frac{1}{2}\ln^2(1+y)+\ln(2)\ln\left(\frac{1-y}{1+y}\right).$$

Divide both sides by y then integrate from $y=0$ to x using $\int_0^x y^{n-1}\mathrm{d}y=\frac{x^n}{n}$,

$$\sum_{n=1}^{\infty}\frac{H_{\frac{n}{2}}}{n^2}x^n-\sum_{n=1}^{\infty}\frac{\overline{H}_n}{n^2}x^n$$
$$=2\operatorname{Li}_3(x)+\operatorname{Li}_3(-x)+\frac{1}{2}\int_0^x\frac{\ln^2(1-y^2)}{y}\mathrm{d}y-\frac{1}{2}\int_0^x\frac{\ln^2(1+y)}{y}\mathrm{d}y$$
$$+\ln(2)[\operatorname{Li}_2(-x)-\operatorname{Li}_2(x)].$$

First integral: Substitute $1-y^2=t$,

$$\int_0^x\frac{\ln^2(1-y^2)}{y}\,\mathrm{d}y=\frac{1}{2}\int_{1-x^2}^1\frac{\ln^2(t)}{1-t}\mathrm{d}t$$
$$\{\text{recall the result (2.33)}\}$$
$$=\frac{1}{2}\Big[-\ln^2(t)\ln(1-t)-2\ln(t)\operatorname{Li}_2(t)+2\operatorname{Li}_3(t)\Big]_{1-x^2}^1$$
$$=\zeta(3)+\ln(x)\ln^2(1-x^2)+\ln(1-x^2)\operatorname{Li}_2(1-x^2)-\operatorname{Li}_3(1-x^2).$$

Second integral: Substitute $\frac{1}{1+y}=t$,

$$\begin{aligned}\int_0^x \frac{\ln^2(1+y)}{y}\mathrm{d}y &= \int_{\frac{1}{1+x}}^1 \frac{\ln^2(t)}{t(1-t)}\mathrm{d}t\\ &= \int_{\frac{1}{1+x}}^1 \frac{\ln^2(t)}{t}\mathrm{d}t + \int_{\frac{1}{1+x}}^1 \frac{\ln^2(t)}{1-t}\mathrm{d}t\\ &\{\text{recall the result (2.33) for the second integral}\}\\ &= \frac{1}{3}\ln^3(t) - \ln^2(t)\ln(1-t) - 2\ln(t)\operatorname{Li}_2(t) + 2\operatorname{Li}_3(t)\Big|_{\frac{1}{1+x}}^1\\ &= 2\zeta(3) - 2\operatorname{Li}_3\left(\frac{1}{1+x}\right) - 2\ln(1+x)\operatorname{Li}_2\left(\frac{1}{1+x}\right)\\ &\quad + \ln(x)\ln^2(1+x) - \frac{2}{3}\ln^3(1+x).\end{aligned}$$

Collect these two integrals along with (2.31) to finalize the proof.

2.1.24 $\sum_{n=1}^\infty \frac{\binom{2n}{n}}{4^n} H_n x^n$

Let $|x|<1$. Then the following identity holds:

$$\sum_{n=1}^\infty \frac{\binom{2n}{n}}{4^n} H_n x^n = \frac{2}{\sqrt{1-x}} \ln\left(\frac{1+\sqrt{1-x}}{2\sqrt{1-x}}\right). \tag{2.37}$$

The following proof may be found in [1, p. 5]:

Proof. Using the integral form of the harmonic number, we have

$$\begin{aligned}\sum_{n=1}^\infty \frac{\binom{2n}{n}}{4^n} H_n x^n &= \sum_{n=1}^\infty \frac{1}{4^n}\binom{2n}{n}\left(\int_0^1 \frac{1-y^n}{1-y}\mathrm{d}y\right)x^n\\ &= \int_0^1 \frac{1}{1-y}\left(\sum_{n=1}^\infty \frac{\binom{2n}{n}}{4^n}x^n - \sum_{n=1}^\infty \frac{\binom{2n}{n}}{4^n}(xy)^n\right)\mathrm{d}y\\ &\{\text{make use of the Taylor series}\}\\ &= \int_0^1 \frac{1}{1-y}\left(\frac{1}{\sqrt{1-x}} - \frac{1}{\sqrt{1-xy}}\right)\mathrm{d}y\\ &= \frac{1}{\sqrt{1-x}}\int_0^1 \frac{\sqrt{1-xy}-\sqrt{1-x}}{(1-y)\sqrt{1-x}}\mathrm{d}y\end{aligned}$$

$$\begin{aligned}
&\overset{\sqrt{1-xy}=t}{=} \frac{2}{\sqrt{1-x}}\int_{\sqrt{1-x}}^{1}\frac{t-\sqrt{1-x}}{t^2-(1-x)}\mathrm{d}t\\
&= \frac{2}{\sqrt{1-x}}\int_{\sqrt{1-x}}^{1}\frac{\mathrm{d}t}{t+\sqrt{1-x}}\\
&= \frac{2}{\sqrt{1-x}}\ln\left(t+\sqrt{1-x}\right)\Big|_{t=\sqrt{1-x}}^{t=1}\\
&= \frac{2}{\sqrt{1-x}}\left[\ln\left(1+\sqrt{1-x}\right)-\ln\left(2\sqrt{1-x}\right)\right]\\
&= \frac{2}{\sqrt{1-x}}\ln\left(\frac{1+\sqrt{1-x}}{2\sqrt{1-x}}\right).
\end{aligned}$$

2.1.25 $\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{4^n}\frac{H_n}{n}x^n$

Let $|x|\le 1$. Then the following identity holds:

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{4^n}\frac{H_n}{n}x^n = 2\,\mathrm{Li}_2\left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right). \tag{2.38}$$

Proof.

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{4^n}\frac{H_n}{n}x^n &= \sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{4^n}H_n\left(\int x^{n-1}\mathrm{d}x\right)\\
&= \int\frac{1}{x}\left(\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{4^n}H_nx^n\right)\mathrm{d}x
\end{aligned}$$

{recall the generating function (2.37)}

$$\begin{aligned}
&= \int\frac{1}{x}\left(\frac{2}{\sqrt{1-x}}\ln\left(\frac{1+\sqrt{1-x}}{2\sqrt{1-x}}\right)\right)\mathrm{d}x\\
&\overset{\sqrt{1-x}=y}{=} -4\int\frac{\ln\left(\frac{1+y}{2y}\right)}{1-y^2}\mathrm{d}y\\
&\overset{y=\frac{1-t}{1+t}}{=} -2\int\frac{\ln(1-t)}{t}\mathrm{d}t\\
&= 2\,\mathrm{Li}_2(t)
\end{aligned}$$

{substitute $t=(1-y)/(1+y)$ back}

$$= 2\,\mathrm{Li}_2\left(\frac{1-y}{1+y}\right)$$

$$\{\text{substitute } y=\sqrt{1-x} \text{ back}\}$$
$$=2\operatorname{Li}_2\left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)+c.$$

Setting $x=0$ gives $c=0$ and the proof is completed.

2.1.26 $\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{4^n}\frac{H_n}{n^2}x^n$

Let $-1\le x<1$. Then the following identity holds:

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{4^n}\frac{H_n}{n^2}x^n=-4\ln\left(1+\sqrt{\frac{2x}{1+x}}\right)\operatorname{Li}_2\left(\sqrt{\frac{2x}{1+x}}\right)$$
$$+2\operatorname{Li}_3\left(\sqrt{\frac{2x}{1+x}}\right)-4\int_0^{\sqrt{\frac{2x}{1+x}}}\frac{\ln(1-t)\ln(1+t)}{t}\mathrm{d}t. \tag{2.39}$$

Proof.

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{4^n}\frac{H_n}{n^2}x^n=\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{4^n}\frac{H_n}{n}\left(\int_0^x y^{n-1}\mathrm{d}y\right)$$
$$=\int_0^x\frac{1}{y}\left(\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{4^n}\frac{H_n}{n}y^n\right)\mathrm{d}y$$
$$\{\text{recall the generating function (2.38)}\}$$
$$=\int_0^x\frac{1}{y}\left(2\operatorname{Li}_2\left(\frac{1-\sqrt{1-y}}{1+\sqrt{1-y}}\right)\right)\mathrm{d}y$$
$$\overset{\sqrt{1-y}=u}{=}-4\int_1^{\sqrt{1-x}}\frac{u\operatorname{Li}_2\left(\frac{1-u}{1+u}\right)}{1-u^2}\mathrm{d}u$$
$$\overset{\frac{1-u}{1+u}=t}{=}2\int_0^{\sqrt{\frac{2x}{1+x}}}\frac{(1-t)\operatorname{Li}_2(t)}{t(1+t)}\mathrm{d}t$$
$$=2\int_0^{\sqrt{\frac{2x}{1+x}}}\frac{\operatorname{Li}_2(t)}{t}\mathrm{d}t-4\underbrace{\int_0^{\sqrt{\frac{2x}{1+x}}}\frac{\operatorname{Li}_2(t)}{1+t}\mathrm{d}t}_{\text{IBP}}$$
$$=2\operatorname{Li}_3\left(\sqrt{\frac{2x}{1+x}}\right)-4\ln\left(1+\sqrt{\frac{2x}{1+x}}\right)\operatorname{Li}_2\left(\sqrt{\frac{2x}{1+x}}\right)$$
$$-4\int_0^{\sqrt{\frac{2x}{1+x}}}\frac{\ln(1-t)\ln(1+t)}{t}\mathrm{d}t.$$

2.1.27 $\sum_{n=1}^{\infty} \frac{2H_{2n}-H_n}{n} x^{2n}$

Let $|x| < 1$. Then the following identity holds:

$$\sum_{n=1}^{\infty} \frac{2H_{2n}-H_n}{n} x^{2n} = 2\operatorname{arctanh}^2(x). \tag{2.40}$$

Proof. Squaring both sides of $\operatorname{arctanh}(x) = -\frac{1}{2}\ln\left(\frac{1-x}{1+x}\right)$, we get

$$\begin{aligned}
\operatorname{arctanh}^2(x) &= \frac{1}{4}\ln^2\left(\frac{1-x}{1+x}\right)\\
&\left\{\text{use } \frac{1}{4}(a-b)^2 = \frac{1}{2}a^2 + \frac{1}{2}b^2 - \frac{1}{4}(a+b)^2\right\}\\
&\{\text{with } a = \ln(1-x) \text{ and } b = \ln(1+x)\}\\
&= \frac{1}{2}\ln^2(1-x) + \frac{1}{2}\ln^2(1+x) - \frac{1}{4}\ln^2(1-x^2)\\
&\{\text{expand all squared logs in series as in (2.7)}\}\\
&= \sum_{n=1}^{\infty} \frac{H_{n-1}}{n} x^n + \sum_{n=1}^{\infty} (-1)^n \frac{H_{n-1}}{n} x^n - \frac{1}{2}\sum_{n=1}^{\infty} \frac{H_{n-1}}{n} x^{2n}\\
&\left\{\text{use } \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} (-1)^n a_n = 2\sum_{n=1}^{\infty} a_{2n} \text{ for the first two sums}\right\}\\
&= 2\sum_{n=1}^{\infty} \frac{H_{2n-1}}{2n} x^{2n} - \frac{1}{2}\sum_{n=1}^{\infty} \frac{H_{n-1}}{n} x^{2n}\\
&= 2\sum_{n=1}^{\infty} \frac{H_{2n} - \frac{1}{2n}}{2n} x^{2n} - \frac{1}{2}\sum_{n=1}^{\infty} \frac{H_n - \frac{1}{n}}{n} x^{2n}\\
&= \frac{1}{2}\sum_{n=1}^{\infty} \frac{2H_{2n}-H_n}{n} x^{2n}.
\end{aligned}$$

If we replace x with ix in (2.40) then use $\operatorname{arctanh}^2(ix) = -\arctan^2(x)$, we get

$$\sum_{n=1}^{\infty} (-1)^n \frac{2H_{2n}-H_n}{n} x^{2n} = -2\arctan^2(x), \quad |x| \le 1. \tag{2.41}$$

Additionally, by differentiating (2.40) and (2.41) with respect to x, we find

$$\sum_{n=1}^{\infty}(2H_{2n}-H_n)x^{2n-1}=\frac{2\operatorname{arctanh}(x)}{1-x^2}, \tag{2.42}$$

$$\sum_{n=1}^{\infty}(-1)^n(2H_{2n}-H_n)x^{2n-1}=-\frac{2\arctan(x)}{1+x^2}. \tag{2.43}$$

2.1.28 $\sum_{n=1}^{\infty}\frac{H_{2n}}{2n+1}x^{2n+1}$

Let $|x|<1$. Then the following identity holds:

$$\sum_{n=1}^{\infty}\frac{H_{2n}}{2n+1}x^{2n+1}=-\frac{1}{2}\operatorname{arctanh}(x)\ln(1-x^2). \tag{2.44}$$

Proof.

$$\begin{aligned}
\operatorname{arctanh}(x)\ln(1-x^2)&=\frac{1}{2}\{\ln(1+x)-\ln(1-x)\}\{\ln(1+x)+\ln(1-x)\}\\
&=\frac{1}{2}\ln^2(1+x)-\frac{1}{2}\ln^2(1-x)\\
&\quad\{\text{expand both squared logs in series as in (2.7)}\}\\
&=\sum_{n=1}^{\infty}(-1)^n\frac{H_{n-1}}{n}x^n-\sum_{n=1}^{\infty}\frac{H_{n-1}}{n}x^n\\
&\quad\left\{\text{use}\sum_{n=1}^{\infty}(-1)^n a_n-\sum_{n=1}^{\infty}a_n=-2\sum_{n=0}^{\infty}a_{2n+1}\right\}\\
&=-2\sum_{n=0}^{\infty}\frac{H_{2n}}{2n+1}x^{2n+1}\\
&\quad\{\text{let the index start from 1, since } H_0=0\}\\
&=-2\sum_{n=1}^{\infty}\frac{H_{2n}}{2n+1}x^{2n+1}.
\end{aligned}$$

Let's differentiate both sides of (2.44) with respect to x,

$$\sum_{n=1}^{\infty}H_{2n}x^{2n}=\frac{x\operatorname{arctanh}(x)}{1-x^2}-\frac{\ln(1-x^2)}{2(1-x^2)}.$$

2.1.29 $\sum_{n=1}^{\infty} \frac{(-1)^n H_{2n}}{2n+1} x^{2n+1}$

Let $|x| \leq 1$. Then the following identity holds:

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_{2n}}{2n+1} x^{2n+1} = -\frac{1}{2} \arctan(x) \ln(1+x^2). \tag{2.45}$$

Proof. By writing $\frac{x^{2n+1}}{2n+1} = \int_0^x t^{2n} \mathrm{d}t$, we get

$$\begin{aligned}
2\sum_{n=1}^{\infty} \frac{(-1)^n H_{2n}}{2n+1} x^{2n+1} &= 2\sum_{n=1}^{\infty} (-1)^n H_{2n} \left(\int_0^x t^{2n} \mathrm{d}t\right) \\
&\{\text{write } -1 = i^2\} \\
&= \int_0^x \left(2\sum_{n=1}^{\infty} (ix)^{2n} H_{2n}\right) \mathrm{d}t \\
&\left\{\text{use } 2\sum_{n=1}^{\infty} a_{2n} = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} (-1)^n a_n \text{ given in (1.5)}\right\} \\
&= \int_0^x \left(\sum_{n=1}^{\infty} (it)^n H_n + \sum_{n=1}^{\infty} (-it)^n H_n\right) \mathrm{d}t \\
&\{\text{make use of the generating function (2.4)}\} \\
&= \int_0^x \left(-\frac{\ln(1-it)}{1-it} - \frac{\ln(1+it)}{1+it}\right) \mathrm{d}t \\
&= \int_0^x \left(-\frac{\ln(1-it)+\ln(1+it)+it(\ln(1-it)-\ln(1+it))}{1+t^2}\right) \mathrm{d}t \\
&= \int_0^x \left(-\frac{\ln(1+t^2)+2t\arctan t}{1+t^2}\right) \mathrm{d}t \\
&= -\int_0^x \mathrm{d}(\ln(1+t^2)\arctan t) \\
&= -\ln(1+t^2)\arctan t\Big|_0^x \\
&= -\ln(1+x^2)\arctan x.
\end{aligned}$$

For a different proof, replace x with ix in (2.44) then use $\operatorname{arctanh}(ix) = i\arctan(x)$.

2.1.30 $\sum_{n=1}^{\infty}\left(\frac{H_n-H_{2n}}{n}-\frac{1}{2n^2}\right)x^{2n}$

Let $|x|<1$. Then the following identity holds:

$$\sum_{n=1}^{\infty}\left(\frac{H_n-H_{2n}}{n}-\frac{1}{2n^2}\right)x^{2n}=\ln(1-x)\ln(1+x). \tag{2.46}$$

Proof (i). Put $a=\ln(1-x)$ and $b=\ln(1+x)$ in the algebraic identity:

$$ab=\frac{1}{4}(a+b)^2-\frac{1}{4}(a-b)^2,$$

we have

$$\begin{aligned}\ln(1-x)\ln(1+x)&=\frac{1}{4}\ln^2(1-x^2)-\frac{1}{4}\ln^2\left(\frac{1-x}{1+x}\right)\\&=\frac{1}{4}\ln^2(1-x^2)-\operatorname{arctanh}^2(x)\\&\qquad\{\text{expand the first squared log in series as in (2.7)}\}\\&\qquad\left\{\text{and substitute the result of }\operatorname{arctanh}^2(x)\text{ given in (2.40)}\right\}\\&=\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_{n-1}}{n}x^{2n}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{2H_{2n}-H_n}{n}x^{2n}\\&=\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n-\frac{1}{n}}{n}x^{2n}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{2H_{2n}-H_n}{n}x^{2n}\\&=\sum_{n=1}^{\infty}\left(\frac{H_n-H_{2n}}{n}-\frac{1}{2n^2}\right)x^{2n}.\end{aligned}$$

Proof (ii). Differentiate $\ln(1-x)\ln(1+x)$ then integrate back,

$$\begin{aligned}\ln(1-x)\ln(1+x)&=\int \mathrm{d}(\ln(1-x)\ln(1+x))\\&=\int\frac{\ln(1-x)}{1+x}\mathrm{d}x-\int\frac{\ln(1+x)}{1-x}\mathrm{d}x\\&\qquad\{\text{recall the generating function (2.29)}\}\\&=\int\left(\sum_{n=1}^{\infty}(-1)^n\overline{H}_nx^n\right)\mathrm{d}x-\int\left(\sum_{n=1}^{\infty}\overline{H}_nx^n\right)\mathrm{d}x\\&=\sum_{n=1}^{\infty}(-1)^n\overline{H}_n\left(\int x^n\mathrm{d}x\right)-\sum_{n=1}^{\infty}\overline{H}_n\left(\int x^n\mathrm{d}x\right)\end{aligned}$$

$$= \sum_{n=1}^{\infty} (-1)^n \overline{H}_n \left(\frac{x^{n+1}}{n+1}\right) - \sum_{n=1}^{\infty} \overline{H}_n \left(\frac{x^{n+1}}{n+1}\right)$$

$$\left\{\text{use} \sum_{n=1}^{\infty} (-1)^n a_n - \sum_{n=1}^{\infty} a_n = -2 \sum_{n=1}^{\infty} a_{2n-1} \text{ given in (1.9)}\right\}$$

$$= -2 \sum_{n=1}^{\infty} \overline{H}_{2n-1} \frac{x^{2n}}{2n}$$

$$\left\{\text{employ } \overline{H}_{n-1} = \overline{H}_n + \frac{(-1)^n}{n} \text{ given in (1.136)}\right\}$$

$$= -2 \sum_{n=1}^{\infty} \left(\overline{H}_{2n} + \frac{1}{2n}\right) \frac{x^{2n}}{2n}$$

$$\{\text{recall the relation (1.137)}\}$$

$$= -2 \sum_{n=1}^{\infty} \left(H_{2n} - H_n + \frac{1}{2n}\right) \frac{x^{2n}}{2n}$$

$$= \sum_{n=1}^{\infty} \left(\frac{H_n - H_{2n}}{n} - \frac{1}{2n^2}\right) x^{2n} + c.$$

By setting $x = 0$, we find $c = 0$ and the proof is complete.
For a different proof, see [32, p. 334].

2.2 Series Expansion of Powers of $\arcsin(z)$

2.2.1 Series Expansion of $\arcsin(z)$

Let $|z| \le 1$. Then the following identity holds:

$$\arcsin(z) = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n} \frac{z^{2n+1}}{2n+1}. \tag{2.47}$$

Proof. Differentiate $\arcsin(z)$ then integrate,

$$\arcsin(z) = \int \mathrm{d}(\arcsin(z))$$

$$= \int \frac{1}{\sqrt{1-z^2}} \mathrm{d}z$$

$$\left\{\text{expand } \frac{1}{\sqrt{1-z^2}} \text{ in Taylor series as } \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n} z^{2n}\right\}$$

$$
\begin{aligned}
&= \int \left(\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n} z^{2n} \right) \mathrm{d}z \\
&= \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n} \left(\int z^{2n} \mathrm{d}z \right) \\
&= \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n} \frac{z^{2n+1}}{2n+1} + c.
\end{aligned}
$$

Extracting $c = 0$ by setting $z = 0$ completes the proof.

2.2.2 Series Expansion of $\frac{\arcsin(z)}{\sqrt{1-z^2}}$

Let $|z| < 1$. Then the following identity holds:

$$\frac{\arcsin(z)}{\sqrt{1-z^2}} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}} \frac{z^{2n-1}}{n}. \tag{2.48}$$

The following proof may be found in [46]:

Proof.

$$
\begin{aligned}
\frac{1}{2} \sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}} \frac{z^{2n-1}}{n} &= \sum_{n=1}^{\infty} z^{2n-1} \left(\frac{4^n}{\binom{2n}{n}} \frac{1}{2n} \right) \\
&\quad \{\text{recall the integral (1.45)}\} \\
&= \sum_{n=1}^{\infty} z^{2n-1} \left(\int_0^{\frac{\pi}{2}} \sin^{2n-1}(x) \mathrm{d}x \right) \\
&= \int_0^{\frac{\pi}{2}} \frac{1}{z \sin(x)} \left(\sum_{n=1}^{\infty} \left(z^2 \sin^2(x)\right)^n \right) \mathrm{d}x \\
&\quad \{\text{employ the geometric series formula}\} \\
&= \int_0^{\frac{\pi}{2}} \frac{1}{z \sin x} \left(\frac{z^2 \sin^2(x)}{1 - z^2 \sin^2(x)} \right) \mathrm{d}x \\
&= \int_0^{\frac{\pi}{2}} \frac{z \sin x}{1 - z^2 + z^2 \cos^2(x)} \mathrm{d}x \\
&= -\frac{1}{\sqrt{1-z^2}} \arctan\left(\frac{z \cos x}{\sqrt{1-z^2}} \right) \Bigg|_{x=0}^{x=\frac{\pi}{2}} \\
&= \frac{1}{\sqrt{1-z^2}} \arctan\left(\frac{z}{\sqrt{1-z^2}} \right)
\end{aligned}
$$

$$= \frac{\arcsin(z)}{\sqrt{1-z^2}}.$$

To justify the last step, differentiate $\arctan\left(\frac{z}{\sqrt{1-z^2}}\right)$ then integrate back.
If we integrate both sides of (2.48) from $z = 0$ to z, we get

$$\arcsin^2(z) = \frac{1}{2}\sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}} \frac{z^{2n}}{n^2}, \quad |z| \le 1. \tag{2.49}$$

2.2.3 Series Expansion of $\arcsin^3(z)$

Let $|z| \le 1$. Then the following identity holds:

$$\arcsin^3(z) = 6\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n} \left(H_{2n}^{(2)} - \frac{1}{4}H_n^{(2)}\right) \frac{z^{2n+1}}{2n+1}. \tag{2.50}$$

The following proof may be found in [46]:
Proof. Let $\arcsin(z) = x$ in (2.47):

$$\arcsin(z) = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n} \frac{z^{2n+1}}{2n+1}$$

and write

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2} = \left(\frac{(2n)!}{n!}\right)^2 \frac{1}{(2n)!},$$

we have

$$x = \sum_{n=0}^{\infty} \left(\frac{(2n)!}{2^n n!}\right)^2 \frac{\sin^{2n+1}(x)}{(2n+1)!} = \sum_{n=0}^{\infty} b_n^2 f_n(x), \tag{2.51}$$

where

$$f_n(x) = \frac{\sin^{2n+1}(x)}{(2n+1)!}, \quad b_n = \frac{(2n)!}{2^n n!}.$$

Note that

$$f_n''(x) = f_n(x) - (2n+1)^2 f_n(x),$$
$$b_{n+1} = (2n+1)b_n.$$

In light of (2.51), write

$$x^3 = \sum_{n=0}^{\infty} a_n b_n^2 f_n(x). \tag{2.52}$$

Assuming $a_0 = 0$ allows the index n to start from 1,

$$x^3 = \sum_{n=1}^{\infty} a_n b_n^2 f_n(x).$$

Differentiate twice with respect to x to get

$$\begin{aligned}
x &= \frac{1}{6} \sum_{n=1}^{\infty} a_n b_n^2 f_n''(x) \\
&\qquad \{\text{substitute } f_n''(x)\} \\
&= \frac{1}{6} \sum_{n=1}^{\infty} a_n b_n^2 f_{n-1}(x) - \frac{1}{6} \sum_{n=1}^{\infty} (2n+1)^2 a_n b_n^2 f_n(x) \\
&\qquad \{\text{shift the index } n \text{ by } +1 \text{ in the first sum}\} \\
&\qquad \{\text{and let } n \text{ start from 0 in the second sum, since we assumed } a_0 = 0\} \\
&= \frac{1}{6} \sum_{n=0}^{\infty} a_{n+1} b_{n+1}^2 f_n(x) - \frac{1}{6} \sum_{n=0}^{\infty} (2n+1)^2 a_n b_n^2 f_n(x) \\
&\qquad \{\text{substitute } b_{n+1} \text{ in the first sum}\} \\
&= \frac{1}{6} \sum_{n=0}^{\infty} (2n+1)^2 a_{n+1} b_n^2 f_n(x) - \frac{1}{6} \sum_{n=0}^{\infty} (2n+1)^2 a_n b_n^2 f_n(x) \\
&= \sum_{n=0}^{\infty} \frac{(2n+1)^2 (a_{n+1} - a_n)}{6} b_n^2 f_n(x). \qquad (2.53)
\end{aligned}$$

By comparing the series in (2.51) and (2.53), we see that

$$1 = \frac{(2n+1)^2 (a_{n+1} - a_n)}{6}$$

or

$$a_{n+1} - a_n = \frac{6}{(2n+1)^2}.$$

Employing the generalization (1.121), we find

$$a_n = 6 \left(H_{2n}^{(2)} - \frac{1}{4} H_n^{(2)} \right).$$

Notice that a_n meets our assumption $(a_0 = 0)$, since $H_0^{(2)} = 0$.

Substitute a_n in (2.52),

$$x^3 = 6 \sum_{n=0}^{\infty} \left(H_{2n}^{(2)} - \frac{1}{4} H_n^{(2)} \right) b_n^2 f_n(x)$$

$$\{\text{substitute } b_n \text{ and } f_n(x)\}$$
$$= 6\sum_{n=0}^{\infty}\left(H_{2n}^{(2)} - \frac{1}{4}H_n^{(2)}\right)\left(\frac{(2n)!}{2^n n!}\right)^2 \frac{\sin^{2n+1}(x)}{(2n+1)!}$$
$$\left\{\text{write } (n!)^2 = (2n)!\binom{2n}{n}\right\}$$
$$= 6\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{4^n}\left(H_{2n}^{(2)} - \frac{1}{4}H_n^{(2)}\right)\frac{\sin^{2n+1}(x)}{2n+1}.$$

Let $x = \arcsin(z)$ to finish the proof.

2.2.4 Series Expansion of $\arcsin^4(z)$

Let $|z| \leq 1$. Then the following identity holds:

$$\arcsin^4(z) = \frac{3}{2}\sum_{n=1}^{\infty}\frac{4^n}{\binom{2n}{n}}\frac{H_{n-1}^{(2)}z^{2n}}{n^2}. \tag{2.54}$$

The following proof may be found in [46]:

Proof. Set $\arcsin(z) = x$ in (2.49):

$$\arcsin^2(z) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{4^n}{\binom{2n}{n}}\frac{z^{2n}}{n^2}$$

and write

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2} = \frac{(2n)!}{(n(n-1)!)^2},$$

we get

$$x^2 = \frac{1}{2}\sum_{n=1}^{\infty}(2^n(n-1)!)^2\frac{\sin^{2n}(x)}{(2n)!} = \frac{1}{2}\sum_{n=1}^{\infty}b_n^2 f_n(x), \tag{2.55}$$

where

$$f_n(x) = \frac{\sin^{2n}(x)}{(2n)!}, \quad b_n = 2^n(n-1)!.$$

Note that

$$f_n''(x) = f_{n-1}(x) - (2n)^2 f_n(x),$$
$$b_{n+1} = 2nb_n.$$

In light of (2.55), write

$$x^4 = \frac{1}{2}\sum_{n=1}^{\infty} a_n b_n^2 f_n(x). \tag{2.56}$$

Assuming $a_1 = 0$ allows the index n to start from 2,

$$x^4 = \frac{1}{2}\sum_{n=2}^{\infty} a_n b_n^2 f_n(x).$$

Upon differentiating twice with respect to x, we get

$$\begin{aligned}
x^2 &= \frac{1}{24}\sum_{n=2}^{\infty} a_n b_n^2 f_n''(x) \\
&\qquad \{\text{substitute } f_n''(x)\} \\
&= \frac{1}{24}\sum_{n=2}^{\infty} a_n b_n^2 f_{n-1}(x) - \frac{1}{24}\sum_{n=2}^{\infty} a_n (2n)^2 b_n^2 f_n(x) \\
&\qquad \{\text{shift the index } n \text{ of the first sum by } +1\ \} \\
&\qquad \{\text{and let } n \text{ start from } 1 \text{ in the second sum, since we assumed } a_1 = 0\} \\
&= \frac{1}{24}\sum_{n=1}^{\infty} a_{n+1} b_{n+1}^2 f_n(x) - \frac{1}{24}\sum_{n=1}^{\infty} a_n (2n)^2 b_n^2 f_n(x) \\
&\qquad \{\text{substitute } b_{n+1} \text{ in the first sum}\} \\
&= \frac{1}{24}\sum_{n=1}^{\infty} (2n)^2 a_{n+1} b_n^2 f_n(x) - \frac{1}{24}\sum_{n=1}^{\infty} (2n)^2 a_n b_n^2 f_n(x) \\
&= \frac{1}{2}\sum_{n=1}^{\infty} \frac{(2n)^2 (a_{n+1} - a_n)}{12} b_n^2 f_n(x).
\end{aligned} \tag{2.57}$$

By comparing the coefficients of $f_n(x)$ in (2.55) and (2.57), we see that

$$1 = \frac{(2n)^2 (a_{n+1} - a_n)}{12}$$

or

$$a_{n+1} - a_n = \frac{3}{n^2}.$$

Using (1.117), we find

$$a_n = 3H_{n-1}^{(2)}.$$

Notice that a_n meets our assumption $(a_1 = 0)$.

Substitute a_n in (2.56), we obtain

$$\begin{aligned}
x^4 &= \frac{3}{2}\sum_{n=1}^{\infty} H_{n-1}^{(2)} b_n^2 f_n(x) \\
&\qquad \{\text{substitute } b_n \text{ and } f_n(x)\} \\
&= \frac{3}{2}\sum_{n=1}^{\infty} H_{n-1}^{(2)} (2^n(n-1)!)^2 \frac{\sin^{2n}(x)}{(2n)!} \\
&\qquad \left\{\text{write } (2n)! = (n(n-1)!)^2 \binom{2n}{n}\right\} \\
&= \frac{3}{2}\sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}} \frac{H_{n-1}^{(2)} \sin^{2n}(x)}{n^2}.
\end{aligned}$$

Let $x = \arcsin(z)$ to complete the proof.
By differentiating both sides of (2.50) and (2.54), we obtain

$$\frac{\arcsin^2(z)}{\sqrt{1-z^2}} = 2\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{4^n} \left(H_{2n}^{(2)} - \frac{1}{4} H_n^{(2)}\right) z^{2n} \quad |z| < 1 \tag{2.58}$$

and

$$\frac{\arcsin^3(z)}{\sqrt{1-z^2}} = \frac{3}{4}\sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}} \frac{H_{n-1}^{(2)} z^{2n-1}}{n}. \quad |z| < 1. \tag{2.59}$$

2.3 Identities by Binomial Coefficient

Let $|x| < 1$ and $a \in \mathbb{Z}^+$. Then the following identity holds:

$$(-1)^a \frac{\ln^a(1-x)}{1-x} = \sum_{n=1}^{\infty} f(a,n) x^n, \tag{2.60}$$

where

$$f(a,n) = (-1)^{a-1}(a-1)! \left(H_n^{(a)} + \sum_{j=1}^{a-1} \frac{(-1)^j}{j!} H_n^{(a-j)} f(j,n) \right).$$

Proof.

$$(-1)^a \frac{\ln^a(1-x)}{1-x} = \lim_{m\to 1} (-1)^a \ln^a(1-x)(1-x)^{-m}$$

$$= \lim_{m\to 1} \frac{\partial^a}{\partial m^a}(1-x)^{-m}$$

$$\left\{\text{expand } (1-x)^{-m} \text{ in Taylor series}\right\}$$

$$= \lim_{m\to 1} \frac{\partial^a}{\partial m^a} \sum_{n=0}^{\infty} \binom{m+n-1}{n} x^n$$

$$\left\{\text{let the index start from one, since } \frac{\partial^a}{\partial m^a}\binom{m-1}{0} = 0\right\}$$

$$= \sum_{n=1}^{\infty} \lim_{m\to 1} \frac{\partial^a}{\partial m^a} \binom{m+n-1}{n} x^n$$

$$= \sum_{n=1}^{\infty} f(a,n) x^n.$$

Let's find the derivative of the binomial coefficient,

$$\frac{\partial}{\partial m}\binom{m+n-1}{n} = \frac{\partial}{\partial m}\frac{\Gamma(m+n)}{\Gamma(n+1)\Gamma(m)}$$

$$\left\{\text{use } \frac{d}{dx}\Gamma(x) = \Gamma(x)\psi(x)\right\}$$

$$= \frac{\Gamma(m)\Gamma(m+n)\psi(m+n) - \Gamma(m+n)\Gamma(m)\psi(m)}{\Gamma(n+1)\Gamma^2(m)}$$

$$= \frac{\Gamma(m+n)}{\Gamma(n+1)\Gamma(m)}\left(\psi(m+n) - \psi(m)\right)$$

$$= \binom{m+n-1}{n}\left(\psi(m+n) - \psi(m)\right)$$

$$\left\{\text{use } \psi(x) = H_{x-1} - \gamma = \sum_{k=1}^{x-1}\frac{1}{k} - \gamma\right\}$$

$$= \binom{m+n-1}{n}\left(\sum_{k=1}^{m+n-1}\frac{1}{k} - \sum_{k=1}^{m-1}\frac{1}{k}\right)$$

$$\left\{\text{use } \sum_{k=1}^{a} f_k = \sum_{k=1}^{b-1} f_k + \sum_{k=b}^{a} f_k,\ a > b \text{ for the first sum}\right\}$$

$$= \binom{m+n-1}{n}\left(\sum_{k=1}^{m-1}\frac{1}{k} + \sum_{k=m}^{m+n-1}\frac{1}{k} - \sum_{k=1}^{m-1}\frac{1}{k}\right)$$

$$= \binom{m+n-1}{n}\left(\sum_{k=m}^{m+n-1}\frac{1}{k}\right)$$

$$\{\text{shift the index by } -m\}$$

$$= \sum_{k=0}^{n-1} \frac{\binom{m+n-1}{n}}{k+m}.$$

Keep differentiating with respect to m:

$$\frac{\partial^2}{\partial m^2}\binom{m+n-1}{n} = -\sum_{k=0}^{n-1} \frac{\binom{m+n-1}{n}}{(k+m)^2} + \sum_{k=0}^{n-1} \frac{\frac{\partial}{\partial m}\binom{m+n-1}{n}}{k+m},$$

$$\begin{aligned}\frac{\partial^3}{\partial m^3}\binom{m+n-1}{n} &= 2\sum_{k=0}^{n-1} \frac{\binom{m+n-1}{n}}{(k+m)^3} - 2\sum_{k=0}^{n-1} \frac{\frac{\partial}{\partial m}\binom{m+n-1}{n}}{(k+m)^2} \\ &\quad + \sum_{k=0}^{n-1} \frac{\frac{\partial^2}{\partial m^2}\binom{m+n-1}{n}}{k+m}.\end{aligned}$$

Observing these derivatives, we can generalize the a-th derivative to

$$\begin{aligned}&\frac{\partial^a}{\partial m^a}\binom{m+n-1}{n} \\ &= (-1)^{a-1}(a-1)!\left(\sum_{k=0}^{n-1} \frac{\binom{m+n-1}{n}}{(k+m)^a} + \sum_{j=1}^{a-1} \frac{(-1)^j}{j!} \sum_{k=0}^{n-1} \frac{\frac{\partial^j}{\partial m^j}\binom{m+n-1}{n}}{(k+m)^{a-j}}\right). \qquad (2.61)\end{aligned}$$

Let $m \to 1$ and remember $\lim_{m\to 1} \frac{\partial^a}{\partial m^a}\binom{m+n-1}{n} = f(a,n)$ observing that

$$\lim_{m\to 1} \sum_{k=0}^{n-1} \frac{1}{(k+m)^r} = \sum_{k=0}^{n-1} \frac{1}{(k+1)^r} = \sum_{k=1}^{n} \frac{1}{k^r} = H_n^{(r)}$$

and

$$\lim_{m\to 1} \binom{m+n-1}{n} = \binom{n}{n} = 1,$$

we get

$$f(a,n) = (-1)^{a-1}(a-1)!\left(H_n^{(a)} + \sum_{j=1}^{a-1} \frac{(-1)^j}{j!} H_n^{(a-j)} f(j,n)\right)$$

and the proof is complete.

Examples:

$$-\frac{\ln(1-x)}{1-x}=\sum_{n=1}^{\infty}H_nx^n;$$

$$\frac{\ln^2(1-x)}{1-x}=\sum_{n=1}^{\infty}(H_n^2-H_n^{(2)})x^n;$$

$$-\frac{\ln^3(1-x)}{1-x}=\sum_{n=1}^{\infty}\left(H_n^3-3H_nH_n^{(2)}+2H_n^{(3)}\right)x^n;$$

$$\frac{\ln^4(1-x)}{1-x}=\sum_{n=1}^{\infty}\left(H_n^4-6H_n^2H_n^{(2)}+8H_nH_n^{(3)}+3\left(H_n^{(2)}\right)^2-6H_n^{(4)}\right)x^n.$$

2.4 Identities by Beta Function

Let $n, a \in \mathbb{Z}^+$. Then the following identity holds:

$$\int_0^1 x^{n-1}\ln^a(1-x)\mathrm{d}x=t(a,n), \tag{2.62}$$

where

$$t(a,n)=(-1)^a(a-1)!\left(\frac{H_n^{(a)}}{n}+\sum_{j=1}^{a-1}\frac{(-1)^j}{j!}H_n^{(a-j)}t(j,n)\right).$$

Proof. Let $x \to 1-x$,

$$\begin{aligned}\int_0^1 x^{n-1}\ln^a(1-x)\mathrm{d}x&=\int_0^1 \ln^a(x)(1-x)^{n-1}\mathrm{d}x\\&=\lim_{m\to1}\int_0^1 x^{m-1}\ln^a(x)(1-x)^{n-1}\mathrm{d}x\\&=\lim_{m\to1}\int_0^1\frac{\partial^a}{\partial m^a}x^{m-1}(1-x)^{n-1}\mathrm{d}x\\&=\lim_{m\to1}\frac{\partial^a}{\partial m^a}\int_0^1 x^{m-1}(1-x)^{n-1}\mathrm{d}x\\&\qquad\{\text{recall the definition of the beta function (1.46)}\}\\&=\lim_{m\to1}\frac{\partial^a}{\partial m^a}\,\mathrm{B}(m,n)\\&=t(a,n).\end{aligned}$$

Let's find the derivative of the beta function,

$$\begin{aligned}\frac{\partial}{\partial m}\,\mathrm{B}(m,n) &= \frac{\partial}{\partial m}\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\\ =\Gamma(m)&\frac{\Gamma(m+n)\Gamma(n)\psi(n)-\Gamma(m+n)\psi(m+n)\Gamma(n)}{\Gamma^2(m+n)}\\ &=-\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}(\psi(m+n)-\psi(n))\\ &=-\,\mathrm{B}(m,n)(\psi(m+n)-\psi(n))\\ &=-\sum_{k=0}^{n-1}\frac{\mathrm{B}(m,n)}{k+m}.\end{aligned}$$

The a-th derivative can be generalized exactly the same way as in (2.61) to

$$\frac{\partial^a}{\partial m^a}\,\mathrm{B}(m,n)$$
$$=(-1)^a(a-1)!\left(\sum_{k=0}^{n-1}\frac{\mathrm{B}(m,n)}{(k+m)^a}+\sum_{j=1}^{a-1}\frac{(-1)^{a-j}}{j!}\sum_{k=0}^{n-1}\frac{\frac{\partial^j}{\partial m^j}\,\mathrm{B}(m,n)}{(k+m)^{a-j}}\right).$$

Let $m\to 1$ and remember $\lim\limits_{m\to 1}\frac{\partial^a}{\partial m^a}\,\mathrm{B}(m,n)=t(a,n)$ observing that

$$\lim_{m\to 1}\sum_{k=0}^{n-1}\frac{1}{(k+m)^r}=H_n^{(r)}$$

and

$$\lim_{m\to 1}\mathrm{B}(m,n)=B(1,n)=\frac{\Gamma(1)\Gamma(n)}{\Gamma(n+1)}=\frac{\Gamma(n)}{n\Gamma(n)}=\frac{1}{n},$$

we get

$$t(a,n)=(-1)^a(a-1)!\left(\frac{H_n^{(a)}}{n}+\sum_{j=1}^{a-1}\frac{(-1)^j}{j!}H_n^{(a-j)}t(j,n)\right)$$

and the proof is complete.

Remark: In the proof above, we used

$$\frac{\partial^a}{\partial m^a}\int_p^q f(x,m,n)\mathrm{d}x=\int_p^q\frac{\partial^a}{\partial m^a}f(x,m,n)\mathrm{d}x, \tag{2.63}$$

which follows from the special case of Leibniz's integral rule (see [48]) is differentiation under the integral sign theorem:

$$\frac{d^a}{dm^a}\int_p^q f(x,m)\mathrm{d}x=\int_p^q \frac{\partial^a}{\partial m^a} f(x,m)\mathrm{d}x, \tag{2.64}$$

Examples:

$$\int_0^1 x^{n-1}\ln(1-x)\mathrm{d}x=-\frac{H_n}{n};$$

$$\int_0^1 x^{n-1}\ln^2(1-x)\mathrm{d}x=\frac{H_n^2+H_n^{(2)}}{n};$$

$$\int_0^1 x^{n-1}\ln^3(1-x)\mathrm{d}x=-\frac{H_n^3+3H_nH_n^{(2)}+2H_n^{(3)}}{n};$$

$$\int_0^1 x^{n-1}\ln^4(1-x)\mathrm{d}x=\frac{H_n^4+6H_n^2H_n^{(2)}+8H_nH_n^{(3)}+3(H_n^{(2)})^2+6H_n^{(4)}}{n}.$$

The first identity is also proved in (1.124).

2.5 Identities by Cauchy Product

2.5.1 Cauchy Product of Two Power Series

Let $\sum_{n=1}^{\infty} a_n x^n$ and $\sum_{n=1}^{\infty} b_n x^n$ be two power series. The Cauchy product of these two series is given by

$$\left(\sum_{n=1}^{\infty} a_n x^n\right)\left(\sum_{n=1}^{\infty} b_n x^n\right)=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} a_k b_{n-k+1}\right)x^{n+1}. \tag{2.65}$$

Proof.

$$\begin{aligned}
&\left(\sum_{n=1}^{\infty} a_n x^n\right)\left(\sum_{n=1}^{\infty} b_n x^n\right)\\
&=\left(a_1x+a_2x^2+a_3x^3+\cdots\right)\left(b_1x+b_2x^2+b_3x^3+\cdots\right)\\
&=(a_1b_1)x^2+\left(a_1b_2+a_2b_1\right)x^3+\left(a_1b_3+a_2b_2+a_3b_1\right)x^4+\cdots\\
&=\left(\sum_{k=1}^{1} a_k b_{2-k}\right)x^2+\left(\sum_{k=1}^{2} a_k b_{3-k}\right)x^3+\left(\sum_{k=1}^{3} a_k b_{4-k}\right)x^4+\cdots
\end{aligned}$$

$$= \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} a_k b_{n-k+1} \right) x^{n+1}.$$

Applying the steps above also gives

$$\left(\sum_{n=0}^{\infty} a_n x^n \right) \left(\sum_{n=0}^{\infty} b_n x^n \right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n, \tag{2.66}$$

$$\left(\sum_{n=0}^{\infty} a_{2n} x^{2n} \right) \left(\sum_{n=0}^{\infty} b_n x^n \right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} a_{2k} b_{n-2k} \right) x^n. \tag{2.67}$$

2.5.2 Cauchy Product of $-\ln(1-x)\operatorname{Li}_2(x)$

Let $-1 \le x < 1$. Then the following identity holds:

$$-\ln(1-x)\operatorname{Li}_2(x) = 2\sum_{n=1}^{\infty} \frac{H_n}{n^2} x^n + \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n} x^n - 3\operatorname{Li}_3(x). \tag{2.68}$$

We apply the same approach as in [32, p. 516]:
Proof. Expand $\operatorname{Li}_2(x)$ and $\ln(1-x)$ in series,

$$\begin{aligned}
(\operatorname{Li}_2(x))(-\ln(1-x)) &= \left(\sum_{n=1}^{\infty} \frac{x^n}{n^2} \right) \left(\sum_{n=1}^{\infty} \frac{x^n}{n} \right) \\
&\left\{ \text{employ (2.65) where } a_n = \frac{1}{n^2} \text{ and } b_n = \frac{1}{n} \right\} \\
&= \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \frac{1}{k^2(n-k+1)} \right) x^{n+1} \\
&\{\text{use the partial fraction decomposition for the inner sum}\} \\
&= \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \frac{1}{(n+1)k^2} + \frac{1}{(n+1)^2 k} + \frac{1}{(n+1)^2(n-k+1)} \right) x^{n+1} \\
&\left\{ \text{use } \sum_{k=1}^{n} \frac{1}{k^2} = H_n^{(2)} \text{ and } \sum_{k=1}^{n} \frac{1}{n-k+1} = \sum_{k=1}^{n} \frac{1}{k} = H_n \text{ given in (1.3)} \right\} \\
&= \sum_{n=1}^{\infty} \left(\frac{H_n^{(2)}}{(n+1)} + \frac{2H_n}{(n+1)^2} \right) x^{n+1} \\
&\{\text{let the index } n \text{ start from 0, since } H_0^{(2)} = H_0 = 0\}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \left(\frac{H_n^{(2)}}{n+1} + \frac{2H_n}{(n+1)^2} \right) x^{n+1} \\
&\qquad \{\text{shift the index } n \text{ by } -1\} \\
&= \sum_{n=1}^{\infty} \left(\frac{H_{n-1}^{(2)}}{n} + \frac{2H_{n-1}}{n^2} \right) x^{n} \\
&= \sum_{n=1}^{\infty} \left(\frac{H_n^{(2)} - \frac{1}{n^2}}{n} + \frac{2H_n - \frac{2}{n}}{n^2} \right) x^{n} \\
&= \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n} x^n + 2\sum_{n=1}^{\infty} \frac{H_n}{n^2} x^n - 3\sum_{n=1}^{\infty} \frac{x^n}{n^3},
\end{aligned}
$$

where the last sum is $\operatorname{Li}_3(x)$.

2.5.3 Cauchy Product of $\operatorname{Li}_2^2(x)$

Let $|x| \le 1$. Then the following identity holds:

$$
\operatorname{Li}_2^2(x) = 4\sum_{n=1}^{\infty} \frac{H_n}{n^3} x^n + 2\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2} x^n - 6\operatorname{Li}_4(x). \tag{2.69}
$$

Proof. Divide both sides of (2.68):

$$
2\sum_{n=1}^{\infty} \frac{H_n}{n^2} x^n + \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n} x^n - 3\operatorname{Li}_3(x) = -\ln(1-x)\operatorname{Li}_2(x)
$$

by x then integrate using $\int x^{n-1}\mathrm{d}x = \frac{x^n}{n}$,

$$
\begin{aligned}
2\sum_{n=1}^{\infty} \frac{H_n}{n^3} x^n + \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2} x^n - 3\operatorname{Li}_4(x) &= \int \frac{-\ln(1-x)\operatorname{Li}_2(x)}{x}\mathrm{d}x \\
&= \frac{1}{2}\operatorname{Li}_2^2(x) + c.
\end{aligned}
$$

The proof finishes on extracting $c = 0$ by setting $x = 0$.

2.5.4 Cauchy Product of $-\ln(1-x)\operatorname{Li}_3(x)$

Let $-1 \le x < 1$. Then the following identity holds:

$$-\ln(1-x)\operatorname{Li}_3(x) = 2\sum_{n=1}^{\infty}\frac{H_n}{n^3}x^n + \sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^2}x^n + \sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n} - 4\operatorname{Li}_4(x). \tag{2.70}$$

Proof.

$$\begin{aligned}
(\operatorname{Li}_3(x))(-\ln(1-x)) &= \left(\sum_{n=1}^{\infty}\frac{x^n}{n^3}\right)\left(\sum_{n=1}^{\infty}\frac{x^n}{n}\right)\\
&\left\{\text{employ (2.65) where } a_n = \frac{1}{n^3} \text{ and } b_n = \frac{1}{n}\right\}\\
&= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{1}{k^3(n-k+1)}\right)x^{n+1}\\
&\{\text{make use of the partial fraction decomposition for the inner sum}\}\\
&= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{1}{(n+1)k^3} + \frac{1}{(n+1)^2k^2} + \frac{1}{(n+1)^3k}\right.\\
&\qquad \left. + \frac{1}{(n+1)^3(n-k+1)}\right)x^{n+1}\\
&= \sum_{n=1}^{\infty}\left(\frac{H_n^{(3)}}{n+1} + \frac{H_n^{(2)}}{(n+1)^2} + \frac{2H_n}{(n+1)^3}\right)x^{n+1}\\
&\left\{\text{let the index } n \text{ start from 0, since } H_0^{(3)} = H_0^{(2)} = H_0 = 0\right\}\\
&= \sum_{n=0}^{\infty}\left(\frac{H_n^{(3)}}{n+1} + \frac{H_n^{(2)}}{(n+1)^2} + \frac{2H_n}{(n+1)^3}\right)x^{n+1}\\
&\{\text{shift the index } n \text{ by } -1\}\\
&= \sum_{n=1}^{\infty}\left(\frac{H_{n-1}^{(3)}}{n} + \frac{H_{n-1}^{(2)}}{n^2} + \frac{2H_{n-1}}{n^3}\right)x^n\\
&= \sum_{n=1}^{\infty}\left(\frac{H_n^{(3)} - \frac{1}{n^3}}{n} + \frac{H_n^{(2)} - \frac{1}{n^2}}{n} + 2\frac{H_n - \frac{1}{n}}{n^2}\right)x^n\\
&= \sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n}x^n + \sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^2}x^n + 2\sum_{n=1}^{\infty}\frac{H_n}{n^3}x^n - 4\sum_{n=1}^{\infty}\frac{x^n}{n^4},
\end{aligned}$$

where the last sum is $\operatorname{Li}_4(x)$.

2.5.5 Cauchy Product of $\operatorname{Li}_2(x)\operatorname{Li}_3(x)$

Let $|x| \leq 1$. Then the following identity holds:

$$\operatorname{Li}_2(x)\operatorname{Li}_3(x) = 6\sum_{n=1}^{\infty}\frac{H_n}{n^4}x^n + 3\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3}x^n + \sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^2}x^n - 10\operatorname{Li}_5(x). \tag{2.71}$$

Proof (i).

$$\begin{aligned}
\operatorname{Li}_3(x)\operatorname{Li}_2(x) &= \left(\sum_{n=1}^{\infty}\frac{x^n}{n^3}\right)\left(\sum_{n=1}^{\infty}\frac{x^n}{n^2}\right)\\
&= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{1}{k^3(n-k+1)^2}\right)x^{n+1}\\
&= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{1}{(n+1)^2k^3} + \frac{2}{(n+1)^3k^2} + \frac{1}{(n+1)^3(n-k+1)^2}\right.\\
&\qquad \left. + \frac{3}{(n+1)^4k} + \frac{3}{(n+1)^4(n-k+1)}\right)x^{n+1}\\
&= \sum_{n=1}^{\infty}\left(\frac{H_n^{(3)}}{(n+1)^2} + 3\frac{H_n^{(2)}}{(n+1)^3} + 6\frac{H_n}{(n+1)^4}\right)x^{n+1}\\
&= \sum_{n=1}^{\infty}\left(\frac{H_{n-1}^{(3)}}{n^2} + 3\frac{H_{n-1}^{(2)}}{n^3} + 6\frac{H_{n-1}}{n^4}\right)x^n\\
&= \sum_{n=1}^{\infty}\left(\frac{H_n^{(3)} - \frac{1}{n^3}}{n^2} + 3\frac{H_n^{(2)} - \frac{1}{n^2}}{n^3} + 6\frac{H_n - \frac{1}{n}}{n^4}\right)x^n\\
&= \sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^2}x^n + 3\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3}x^n + 6\sum_{n=1}^{\infty}\frac{H_n}{n^4}x^n - 10\sum_{n=1}^{\infty}\frac{x^n}{n^5}.
\end{aligned}$$

The last sum is $\operatorname{Li}_5(x)$.

Proof (ii).

$$\begin{aligned}
\operatorname{Li}_2(x)\operatorname{Li}_3(x) &= \int \mathrm{d}(\operatorname{Li}_2(x)\operatorname{Li}_3(x))\\
&= \int \frac{1}{x}\left(\operatorname{Li}_2^2(x) - \ln(1-x)\operatorname{Li}_3(x)\right)\mathrm{d}x
\end{aligned}$$

$$\{\text{substitute the results (2.69) and (2.70)}\}$$

$$= \int \frac{1}{x}\left(6\sum_{n=1}^{\infty}\frac{H_n}{n^3}x^n + 3\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^2}x^n + \sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n}x^n - 10\,\mathrm{Li}_4(x)\right)\mathrm{d}x$$

$$\{\text{interchange integration and summation}\}$$

$$= \left(6\sum_{n=1}^{\infty}\frac{H_n}{n^3} + 3\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^2} + \sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n}\right)\int_0^1 x^{n-1}\mathrm{d}x - 10\int_0^1 \frac{\mathrm{Li}_4(x)}{x}\mathrm{d}x$$

$$= 6\sum_{n=1}^{\infty}\frac{H_n}{n^4}x^n + 3\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3}x^n + \sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^2} - 10\,\mathrm{Li}_5(x) + c.$$

The proof finalizes on finding $c = 0$ by setting $x = 0$.

2.5.6 Cauchy Product of $\mathrm{Li}_3^2(x)$

Let $|x| \le 1$. Then the following identity holds:

$$\mathrm{Li}_3^2(x) = 12\sum_{n=1}^{\infty}\frac{H_n}{n^5}x^n + 6\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4}x^n + 2\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^3}x^n - 20\,\mathrm{Li}_6(x). \quad (2.72)$$

Proof. Divide both sides of (2.71) by x then integrate,

$$6\sum_{n=1}^{\infty}\frac{H_n}{n^5}x^n + 3\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4}x^n + \sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^3}x^n - 10\,\mathrm{Li}_6(x)$$
$$= \int \frac{\mathrm{Li}_2(x)\,\mathrm{Li}_3(x)}{x}\mathrm{d}x = \frac{1}{2}\,\mathrm{Li}_3^2(x) + c,$$

where $c = 0$ by setting $x = 0$.

2.5.7 Cauchy Product of $-\ln(1-x)\,\mathrm{Li}_4(x)$

Let $-1 \le x < 1$. Then the following identity holds:

$$-\ln(1-x)\,\mathrm{Li}_4(x) = 2\sum_{n=1}^{\infty}\frac{H_n}{n^4}x^n + \sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3}x^n + \sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^2}x^n$$
$$+ \sum_{n=1}^{\infty}\frac{H_n^{(4)}}{n}x^n - 5\,\mathrm{Li}_5(x). \quad (2.73)$$

The following proof may be found in [32, p. 516]:

Proof.

$$\begin{aligned}
(\mathrm{Li}_4(x))(-\ln(1-x)) &= \left(\sum_{n=1}^{\infty}\frac{x^n}{n^4}\right)\left(\sum_{n=1}^{\infty}\frac{x^n}{n}\right)\\
&= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{1}{k^4(n-k+1)}\right)x^{n+1}\\
&= \sum_{n=1}^{\infty}\left(\frac{H_n^{(4)}}{n+1}+\frac{H_n^{(3)}}{(n+1)^2}+\frac{H_n^{(2)}}{(n+1)^3}+2\frac{H_n}{(n+1)^4}\right)x^{n+1}\\
&= \sum_{n=1}^{\infty}\left(\frac{H_{n-1}^{(4)}}{n}+\frac{H_{n-1}^{(3)}}{n^2}+\frac{H_{n-1}^{(2)}}{n^3}+2\frac{H_{n-1}}{n^4}\right)x^{n}\\
&= \sum_{n=1}^{\infty}\left(\frac{H_n^{(4)}-\frac{1}{n^4}}{n}+\frac{H_n^{(3)}-\frac{1}{n^3}}{n^2}+\frac{H_n^{(2)}-\frac{1}{n^2}}{n^3}+2\frac{H_n-\frac{1}{n}}{n^4}\right)x^{n}\\
&= \sum_{n=1}^{\infty}\left(\frac{H_n^{(4)}}{n}+\frac{H_n^{(3)}}{n^2}+\frac{H_n^{(2)}}{n^3}+2\frac{H_n}{n^4}-\frac{5}{n^5}\right)x^{n}\\
&= 2\sum_{n=1}^{\infty}\frac{H_n}{n^4}x^n+\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3}x^n+\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^2}x^n+\sum_{n=1}^{\infty}\frac{H_n^{(4)}}{n}x^n-5\,\mathrm{Li}_5(x)
\end{aligned}$$

and we are done with the proof.

Applying the Cauchy product, we also find, for $|x|\le 1$, the following identities:

$$\begin{aligned}
\mathrm{Li}_2(x)\,\mathrm{Li}_4(x) &= 8\sum_{n=1}^{\infty}\frac{H_n}{n^5}x^n+4\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4}x^n+2\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^3}x^n\\
&\quad+\sum_{n=1}^{\infty}\frac{H_n^{(4)}}{n^2}x^n-15\,\mathrm{Li}_6(x);\\
\mathrm{Li}_3(x)\,\mathrm{Li}_4(x) &= 20\sum_{n=1}^{\infty}\frac{H_n}{n^6}x^n+10\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^5}x^n+4\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^4}x^n\\
&\quad+\sum_{n=1}^{\infty}\frac{H_n^{(4)}}{n^3}x^n-35\,\mathrm{Li}_7(x);\\
\mathrm{Li}_4^2(x) &= 40\sum_{n=1}^{\infty}\frac{H_n}{n^7}x^n+20\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^6}x^n+8\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^5}x^n\\
&\quad+2\sum_{n=1}^{\infty}\frac{H_n^{(4)}}{n^4}x^n-70\,\mathrm{Li}_8(x).
\end{aligned}$$

Note that the third identity follows from dividing both sides of the second by x then integrating.

2.6 Identities by Abel's Summation

2.6.1 Abel's Summation

Given two finite sums $\sum_{k=1}^{n} a_k$ and $\sum_{k=1}^{n} b_k$, define $A_n = \sum_{i=1}^{n} a_i$. Then

$$\sum_{k=m}^{n} a_k b_k = A_n b_n + A_{m-1} b_{m-1} - \sum_{k=m}^{n-1} A_k \left(b_{k+1} - b_k\right). \tag{2.74}$$

Proof. By the given sum A_n, one can write

$$A_k = \sum_{i=1}^{k} a_i = a_1 + a_2 + \cdots + a_{k-1} + a_k$$

and so

$$A_{k-1} = \sum_{i=1}^{k-1} a_i = a_1 + a_2 + \cdots + a_{k-1}.$$

Subtracting the two sums yields

$$a_k = A_k - A_{k-1}.$$

Multiply both sides by b_k then take the summation from $k = m$ to n,

$$\begin{aligned}\sum_{k=m}^{n} a_k b_k &= \sum_{k=m}^{n} \left(A_k - A_{k-1}\right) b_k \\ &= \sum_{k=m}^{n} A_k b_k - \sum_{k=m}^{n} A_{k-1} b_k.\end{aligned}$$

For the first sum, use the fact that $\displaystyle\sum_{k=m}^{n} f(k) = f(n) + \sum_{k=m}^{n-1} f(k)$,

$$\sum_{k=m}^{n} A_k b_k = A_n b_n + \sum_{k=m}^{n-1} A_k b_k.$$

for the second sum, use $\sum_{k=m}^{n} f(k) = f(m) + \sum_{k=m+1}^{n} f(k)$,

$$\begin{aligned}\sum_{k=m}^{n} A_{k-1}b_k &= A_{m-1}b_m + \sum_{k=m+1}^{n} A_{k-1}b_k \\ &\qquad \{\text{shift the index } k \text{ by } +1\} \\ &= A_{m-1}b_m + \sum_{k=m}^{n-1} A_k b_{k+1}.\end{aligned}$$

Combining the two sums, we obtain

$$\begin{aligned}\sum_{k=m}^{n} a_k b_k &= A_n b_n - A_{m-1}b_m + \sum_{k=m}^{n-1} A_k b_k - \sum_{k=m}^{n-1} A_k b_{k+1} \\ &= A_n b_n - A_{m-1}b_m - \sum_{k=m}^{n-1} A_k \left(b_{k+1} - b_k\right)\end{aligned}$$

and the proof is finished.

Notice that $A_{m-1} = 0$ when $m = 0, 1$. Thus, (2.74) can be written as

$$\begin{aligned}\sum_{k=0}^{n} a_k b_k &= A_n b_n - \sum_{k=0}^{n-1} A_k \left(b_{k+1} - b_k\right), \\ \sum_{k=1}^{n} a_k b_k &= A_n b_n - \sum_{k=1}^{n-1} A_k \left(b_{k+1} - b_k\right).\end{aligned}$$

Also notice that the index k in the RHS of the last identity can start from 0, since $A_0 = 0$. Therefore,

$$\sum_{k=1}^{n} a_k b_k = A_n b_n - \sum_{k=0}^{n-1} A_k \left(b_{k+1} - b_k\right), \quad A_n = \sum_{i=1}^{n} a_i. \tag{2.75}$$

You may find in [6, Theorem 2.20, p. 55] a proof for a similar formula:

$$\sum_{k=m}^{n} a_k b_k = A_n b_{n+1} + A_{m-1}b_m - \sum_{k=m}^{n} A_k \left(b_{k+1} - b_k\right).$$

2.6.2 First Application

Let $p, q \in \mathbb{Z}_{>1}$. Then the following identity holds:

$$\sum_{k=1}^{\infty} \frac{H_k^{(p)}}{k^q} + \sum_{k=1}^{\infty} \frac{H_k^{(q)}}{k^p} = \zeta(p)\zeta(q) + \zeta(p+q). \tag{2.76}$$

Proof. Let $a_k = \frac{1}{k^q}$ and $b_k = H_k^{(p)}$ in (2.75),

$$\begin{aligned}\sum_{k=1}^{n} \frac{H_k^{(p)}}{k^q} &= \left(\sum_{i=1}^{n} \frac{1}{i^q}\right) H_n^{(p)} - \sum_{k=0}^{n-1} \left(\sum_{i=1}^{k} \frac{1}{i^q}\right) \left(H_{k+1}^{(p)} - H_k^{(p)}\right) \\ &\left\{\text{use } H_{k+1}^{(p)} = H_k^{(p)} + \frac{1}{(k+1)^p} \text{ given in (1.117)}\right\} \\ &= H_n^{(q)} H_n^{(p)} - \sum_{k=0}^{n-1} \left(H_k^{(q)}\right) \left(\frac{1}{(k+1)^p}\right) \\ &\{\text{shift the index } k \text{ by } -1\} \\ &= H_n^{(q)} H_n^{(p)} - \sum_{k=1}^{n} \frac{H_{k-1}^{(q)}}{k^p} \\ &= H_n^{(q)} H_n^{(p)} - \sum_{k=1}^{n} \left(\frac{H_k^{(q)}}{k^p} - \frac{1}{k^{p+q}}\right) \\ &= H_n^{(q)} H_n^{(p)} + \zeta(q+p) - \sum_{k=1}^{n} \frac{H_k^{(q)}}{k^p}.\end{aligned}$$

Reorganize the terms, we have

$$\sum_{k=1}^{n} \frac{H_k^{(p)}}{k^q} + \sum_{k=1}^{n} \frac{H_k^{(q)}}{k^p} = H_n^{(q)} H_n^{(p)} + \zeta(q+p).$$

Next, take the limit on both sides letting $n \to \infty$,

$$\sum_{k=1}^{\infty} \frac{H_k^{(p)}}{k^q} + \sum_{k=1}^{\infty} \frac{H_k^{(q)}}{k^p} = \lim_{n\to\infty} \left\{H_n^{(q)} H_n^{(p)} + \zeta(q+p)\right\}.$$

The proof follows on writing

$$\lim_{n\to\infty} H_n^{(a)} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k^a} = \sum_{k=1}^{\infty} \frac{1}{k^a} = \zeta(a).$$

For a different approach, see [32, p. 358].
Setting $q = p$ in (2.76) yields

$$\sum_{k=1}^{\infty} \frac{H_k^{(p)}}{k^p} = \frac{\zeta^2(p) + \zeta(2p)}{2}. \tag{2.77}$$

Examples:

$$\sum_{k=1}^{\infty} \frac{H_k^{(2)}}{k^2} = \frac{\zeta^2(2) + \zeta(4)}{2} = \frac{7}{4}\zeta(4);$$

$$\sum_{k=1}^{\infty} \frac{H_k^{(3)}}{k^3} = \frac{\zeta^2(3) + \zeta(6)}{2};$$

$$\sum_{k=1}^{\infty} \frac{H_k^{(4)}}{k^4} = \frac{\zeta^2(4) + \zeta(8)}{2} = \frac{13}{12}\zeta(8),$$

2.6.3 Second Application

Let $p \in \mathbb{Z}_{>1}$. Then the following identity holds:

$$\sum_{k=1}^{\infty} \frac{\left(H_k^{(p)}\right)^2}{k^p} - \sum_{k=1}^{\infty} \frac{H_k^{(p)}}{k^{2p}} = \frac{\zeta^3(p) - \zeta(3p)}{3}. \tag{2.78}$$

Proof. Let $a_k = \frac{1}{k^p}$ and $b_k = \left(H_k^{(p)}\right)^2$ in (2.75),

$$\sum_{k=1}^{n} \frac{\left(H_k^{(p)}\right)^2}{k^p}$$

$$= \left(\sum_{i=1}^{n} \frac{1}{i^p}\right) \left(H_n^{(p)}\right)^2 - \sum_{k=0}^{n-1} \left(\sum_{i=1}^{k} \frac{1}{i^p}\right) \left(\left(H_{k+1}^{(p)}\right)^2 - \left(H_k^{(p)}\right)^2\right)$$

$$\left\{\text{use } H_{k+1}^{(p)} = H_k^{(p)} + \frac{1}{(k+1)^p} \text{ given in (1.117)}\right\}$$

$$= H_n^{(p)} \left(H_n^{(p)}\right)^2 - \sum_{k=0}^{n-1} H_k^{(p)} \left(\frac{2H_k^{(p)}}{(k+1)^p} + \frac{1}{(k+1)^{2p}}\right)$$

$$\{\text{shift the index } k \text{ by } -1\}$$

$$
\begin{aligned}
&= \left(H_n^{(p)}\right)^3 - \sum_{k=1}^{n} H_{k-1}^{(p)} \left(\frac{2H_{k-1}^{(p)}}{k^p} + \frac{1}{k^{2p}} \right) \\
&= \left(H_n^{(p)}\right)^3 - \sum_{k=1}^{n} \left(H_k^{(p)} - \frac{1}{k^p} \right) \left(\frac{2H_k^{(p)} - \frac{2}{k^p}}{k^p} + \frac{1}{k^{2p}} \right) \\
&= \left(H_n^{(p)}\right)^3 - 2\sum_{k=1}^{n} \frac{\left(H_k^{(p)}\right)^2}{k^p} + 3\sum_{k=1}^{n} \frac{H_k^{(p)}}{k^{2p}} - H_n^{(3p)}.
\end{aligned}
$$

Rearranging the terms,

$$3\sum_{k=1}^{n} \frac{\left(H_k^{(p)}\right)^2}{k^p} - 3\sum_{k=1}^{n} \frac{H_k^{(p)}}{k^{2p}} = \left(H_n^{(p)}\right)^3 - H_n^{(3p)}.$$

Take the limit on both sides letting $n \to \infty$,

$$3\sum_{k=1}^{\infty} \frac{\left(H_k^{(p)}\right)^2}{k^p} - 3\sum_{k=1}^{\infty} \frac{H_k^{(p)}}{k^{2p}} = \lim_{n\to\infty} \left\{ \left(H_n^{(p)}\right)^3 - H_n^{(3p)} \right\}.$$

The proof completes on writing $\lim\limits_{n\to\infty} H_n^{(a)} = \zeta(a)$.

Another approach may be found in [32, p. 359].

Examples:

$$\sum_{k=1}^{\infty} \frac{\left(H_k^{(2)}\right)^2}{k^2} - \sum_{k=1}^{\infty} \frac{H_k^{(2)}}{k^4} = \frac{\zeta^3(2) - \zeta(6)}{3} = \frac{9}{8}\zeta(6);$$

$$\sum_{k=1}^{\infty} \frac{\left(H_k^{(3)}\right)^2}{k^3} - \sum_{k=1}^{\infty} \frac{H_k^{(3)}}{k^6} = \frac{\zeta^3(3) - \zeta(9)}{3};$$

$$\sum_{k=1}^{\infty} \frac{\left(H_k^{(4)}\right)^2}{k^4} - \sum_{k=1}^{\infty} \frac{H_k^{(4)}}{k^8} = \frac{\zeta^3(4) - \zeta(12)}{3} = \frac{493}{5528}\zeta(12),$$

where we used $\zeta^3(2) = \frac{35}{8}\zeta(6)$ and $\zeta^3(4) = \frac{7007}{5528}\zeta(12)$.

2.6.4 Third Application

Let $p, q \in \mathbb{Z}_{>1}$. Then the following identity holds:

$$\sum_{k=1}^{\infty} \frac{H_k^{(q)}}{(2k+1)^p} = \left(1 - \frac{1}{2^p}\right)\zeta(q)\zeta(p) + \left(\frac{1}{2^p} - 2^{q-1}\right)\sum_{k=1}^{\infty}\frac{H_k^{(p)}}{k^q} - 2^{q-1}\sum_{k=1}^{\infty}\frac{(-1)^k H_k^{(p)}}{k^q}. \tag{2.79}$$

Proof. Applying the same technique as in [35], let $a_k = \frac{1}{(2k-1)^p}$ and $b_k = H_k^{(q)} - \zeta(q)$ in (2.75),

$$\sum_{k=1}^{n}\frac{H_k^{(q)} - \zeta(q)}{(2k-1)^p}$$

$$= (H_n^{(q)} - \zeta(q))\sum_{i=1}^{n}\frac{1}{(2i-1)^p} - \sum_{k=0}^{n-1}\left(\sum_{i=1}^{k}\frac{1}{(2i-1)^p}\right)\left(H_{k+1}^{(q)} - H_k^{(q)}\right).$$

Let $n \to \infty$ and write $\lim_{n\to\infty} H_n^{(q)} = \zeta(q)$,

$$\sum_{k=1}^{\infty}\frac{H_k^{(q)} - \zeta(q)}{(2k-1)^p} = 0 - \sum_{k=0}^{\infty}\left(\sum_{i=1}^{k}\frac{1}{(2i-1)^p}\right)\left(H_{k+1}^{(q)} - H_k^{(q)}\right)$$

$$\left\{\text{use} \sum_{i=1}^{k}\frac{1}{(2i-1)^p} = H_{2k}^{(p)} - 2^{-p}H_k^{(p)} \text{ given in (1.120)}\right\}$$

$$\left\{\text{and } H_{k+1}^{(q)} - H_k^{(q)} = \frac{1}{(k+1)^q}\right\}$$

$$= 2^{-p}\sum_{k=0}^{\infty}\frac{H_k^{(p)}}{(k+1)^q} - \sum_{k=0}^{\infty}\frac{H_{2k}^{(p)}}{(k+1)^q}$$

$$\{\text{shift both indexes by } -1\}$$

$$= 2^{-p}\sum_{k=1}^{\infty}\frac{H_{k-1}^{(p)}}{k^q} - \sum_{k=1}^{\infty}\frac{H_{2k-2}^{(p)}}{k^q}$$

$$= 2^{-p}\sum_{k=1}^{\infty}\frac{H_k^{(p)} - \frac{1}{k^p}}{k^q} - \sum_{k=1}^{\infty}\frac{H_{2k}^{(p)} - \frac{1}{(2k)^p} - \frac{1}{(2k-1)^p}}{k^q}$$

$$= 2^{-p}\sum_{k=1}^{\infty}\frac{H_k^{(p)}}{k^q} - \sum_{k=1}^{\infty}\frac{H_{2k}^{(p)}}{k^q} + \sum_{k=1}^{\infty}\frac{1}{k^q(2k-1)^p}$$

$$\{\text{use (1.5) for the second sum}\}$$

$$= 2^{-p}\sum_{k=1}^{\infty}\frac{H_k^{(p)}}{k^q} - 2^{q-1}\sum_{k=1}^{\infty}\frac{H_k^{(p)}}{k^q} - 2^{q-1}\sum_{k=1}^{\infty}\frac{(-1)^k H_k^{(p)}}{k^q} + \sum_{k=1}^{\infty}\frac{1}{k^q(2k-1)^p}$$

$$= \left(2^{-p} - 2^{q-1}\right)\sum_{k=1}^{\infty}\frac{H_k^{(p)}}{k^q} - 2^{q-1}\sum_{k=1}^{\infty}\frac{(-1)^k H_k^{(p)}}{k^q} + \sum_{k=1}^{\infty}\frac{1}{k^q(2k-1)^p}. \tag{2.80}$$

On the other hand,

$$\sum_{k=1}^{\infty}\frac{H_k^{(q)}-\zeta(q)}{(2k-1)^p} = \sum_{k=1}^{\infty}\frac{H_k^{(q)}}{(2k-1)^p} - \zeta(q)\sum_{k=1}^{\infty}\frac{1}{(2k-1)^p}$$

$$\{\text{separate the first term of the first sum}\}$$

$$= 1 + \sum_{k=2}^{\infty}\frac{H_k^{(q)}}{(2k-1)^p} - \zeta(q)\sum_{k=1}^{\infty}\frac{1}{(2k-1)^p}$$

$$\{\text{shift both indexes by } +1\}$$

$$= 1 + \sum_{k=1}^{\infty}\frac{H_{k+1}^{(q)}}{(2k+1)^p} - \zeta(q)\sum_{k=0}^{\infty}\frac{1}{(2k+1)^p}$$

$$= 1 + \sum_{k=1}^{\infty}\frac{H_k^{(q)} + \frac{1}{(k+1)^q}}{(2k+1)^p} - \zeta(q)\sum_{k=0}^{\infty}\frac{1}{(2k+1)^p}$$

$$= 1 + \sum_{k=1}^{\infty}\frac{H_k^{(q)}}{(2k+1)^p} + \sum_{k=1}^{\infty}\frac{1}{(k+1)^q(2k+1)^p} - \zeta(q)\sum_{k=0}^{\infty}\frac{1}{(2k+1)^p}$$

$$\{\text{shift the index of the second sum by } -1\}$$

$$= 1 + \sum_{k=1}^{\infty}\frac{H_k^{(q)}}{(2k+1)^p} + \sum_{k=2}^{\infty}\frac{1}{k^q(2k-1)^p} - \zeta(q)\sum_{k=0}^{\infty}\frac{1}{(2k+1)^p}$$

$$\left\{\text{write} \sum_{k=1}^{\infty}\frac{1}{k^q(2k-1)^p} = 1 + \sum_{k=2}^{\infty}\frac{1}{k^q(2k-1)^p}\right\}$$

$$\{\text{and recall the result of the latter sum (1.65)}\}$$

$$= \sum_{k=1}^{\infty}\frac{H_k^{(q)}}{(2k+1)^p} + \sum_{k=1}^{\infty}\frac{1}{k^q(2k-1)^p} - \zeta(q)(1-2^{-p})\zeta(p). \tag{2.81}$$

Combining (2.80) and (2.81) completes the proof.

2.7 Identities By Fourier Series

2.7.1 Fourier Series

Let $f(x)$ be a function with a period of $2L$ and integrable on the interval $[-L\,,\ L]$. Then its Fourier series is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right], \tag{2.82}$$

where

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\mathrm{d}x, \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x. \end{aligned}$$

Proof. Let $f\left(\frac{Ly}{\pi}\right)$ be integrable on the interval $[-\pi\,,\ \pi]$. Expand it in cosine and sine series,

$$f\left(\frac{Ly}{\pi}\right) = \sum_{n=0}^{\infty} A_n \cos(ny) + \sum_{n=0}^{\infty} B_n \sin(ny),$$

where A_n and B_n are the coefficients of the two series. Separate the first term of both series using $\cos(0) = 1$ and $\sin(0) = 0$,

$$f\left(\frac{Ly}{\pi}\right) = A_0 + \sum_{n=1}^{\infty} A_n \cos(ny) + \sum_{n=1}^{\infty} B_n \sin(ny). \tag{2.83}$$

To find A_0, integrate both sides of (2.83) from $y = -\pi$ to π,

$$\begin{aligned} &\int_{-\pi}^{\pi} f\left(\frac{Ly}{\pi}\right)\mathrm{d}y \\ &= \int_{-\pi}^{\pi} A_0 \mathrm{d}y + \int_{-\pi}^{\pi}\sum_{n=1}^{\infty} A_n \cos(ny)\mathrm{d}y + \int_{-\pi}^{\pi}\sum_{n=1}^{\infty} B_n \sin(ny)\mathrm{d}y \\ &\qquad \{\text{interchange integration and summation}\} \\ &= A_0 y\Big|_{-\pi}^{\pi} + \sum_{n=1}^{\infty} A_n \int_{-\pi}^{\pi}\cos(ny)\mathrm{d}y + \sum_{n=1}^{\infty} B_n \int_{-\pi}^{\pi}\sin(ny)\mathrm{d}y. \end{aligned}$$

Employing the rule:

$$\int_{-a}^{a} f(x)\mathrm{d}x = \begin{cases} 2\int_0^a f(x)\mathrm{d}x & \text{if } f(x) \text{ is even function } (f(-x) = f(x)) \\ 0 & \text{if } f(x) \text{ is odd function } (f(-x) = -f(x)) \end{cases}$$

and the fact that $\cos(ny)$ is an even function and $\sin(ny)$ is an odd function, we get

$$\begin{aligned} \int_{-\pi}^{\pi} f\left(\frac{Ly}{\pi}\right)\mathrm{d}y &= 2\pi A_0 + 2\sum_{n=1}^{\infty} A_n \int_0^{\pi} \cos(ny)\mathrm{d}y + 0 \\ &= 2\pi A_0 + 2\sum_{n=1}^{\infty} A_n \frac{\sin(ny)}{n}\bigg|_0^{\pi} \\ &= 2\pi A_0 + 2\sum_{n=1}^{\infty} A_n \frac{\sin(n\pi)}{n} \\ &\qquad \{\text{note that } \sin(n\pi) = 0 \text{ for integer } n\} \\ &= 2\pi A_0 + 0 \end{aligned}$$

or

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(\frac{Ly}{\pi}\right)\mathrm{d}y.$$

To find A_n, multiply both sides of (2.83) by $\cos(ny)$ then integrate from $-\pi$ to π,

$$\begin{aligned} &\int_{-\pi}^{\pi} f\left(\frac{py}{\pi}\right)\cos(ny)\mathrm{d}y \\ &= \int_{-\pi}^{\pi} A_0\cos(ny)\mathrm{d}y + \sum_{n=1}^{\infty} A_n\int_{-\pi}^{\pi}\cos^2(ny)\mathrm{d}y + \sum_{n=1}^{\infty} B_n\int_{-\pi}^{\pi}\sin(ny)\cos(ny)\mathrm{d}y \\ &\qquad \{\text{the last integral is 0, since the integrand is an odd function}\} \\ &= A_0\frac{\sin(ny)}{n}\bigg|_{-\pi}^{\pi} + \sum_{n=1}^{\infty} A_n \frac{2ny + \sin(2ny)}{4n}\bigg|_{-\pi}^{\pi} \\ &= \frac{2A_0\sin(n\pi)}{n} + \sum_{n=1}^{\infty} A_n\left(\frac{\sin(2n\pi)}{2n} + \pi\right) \\ &= 0 + \pi\sum_{n=1}^{\infty} A_n \end{aligned}$$

or

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{Ly}{\pi}\right)\cos(ny)\mathrm{d}y.$$

To find B_n, multiply both sides of (2.83) by $\sin(ny)$ then integrate from $-\pi$ to π,

$$\int_{-\pi}^{\pi} f\left(\frac{Ly}{\pi}\right)\sin(ny)\mathrm{d}y$$

$$= \int_{-\pi}^{\pi} A_0 \sin(ny)\mathrm{d}y + \sum_{n=1}^{\infty} A_n \int_{-\pi}^{\pi} \cos(ny)\sin(ny)\mathrm{d}y + \sum_{n=1}^{\infty} B_n \int_{-\pi}^{\pi} \sin^2(ny)\mathrm{d}y$$

$\{$the first two integrals are 0, since their integrands are odd functions$\}$

$$= \sum_{n=1}^{\infty} B_n \frac{2ny - \sin(2ny)}{4n}\Bigg|_{-\pi}^{\pi}$$

$$= \sum_{n=1}^{\infty} B_n \left(\pi - \frac{\sin(2n\pi)}{2n}\right)$$

$$= \pi \sum_{n=1}^{\infty} B_n$$

or

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{Ly}{\pi}\right)\sin(ny)\mathrm{d}y.$$

Plugging the results A_0, A_n, and B_n in (2.83) yields

$$f\left(\frac{Ly}{\pi}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(\frac{Ly}{\pi}\right)\mathrm{d}y + \sum_{n=1}^{\infty}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{Ly}{\pi}\right)\cos(ny)\mathrm{d}y\right)\cos(ny)$$
$$+ \sum_{n=1}^{\infty}\left(-\frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{Ly}{\pi}\right)\sin(ny)\mathrm{d}y\right)\sin(ny).$$

Substitute $\frac{Ly}{\pi} = x$,

$$f(x) = \underbrace{\frac{1}{2L}\int_{-L}^{L} f(x)\mathrm{d}x}_{a_o} + \sum_{n=1}^{\infty}\underbrace{\left(\frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x\right)}_{a_n}\cos\left(\frac{n\pi x}{L}\right)$$
$$+ \sum_{n=1}^{\infty}\underbrace{\left(\frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x\right)}_{b_n}\sin\left(\frac{n\pi x}{L}\right)$$
$$= a_0 + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right].$$

Let's replace x with $x + 2L$ to find

$$\cos\left(\frac{n\pi}{L}(x+2L)\right) = \cos\left(\frac{n\pi x}{L} + 2n\pi\right)$$
$$= \cos\left(\frac{n\pi x}{L}\right)\cos(2n\pi) - \sin\left(\frac{n\pi x}{L}\right)\sin(2n\pi)$$
$$\{\text{notice that } \cos(2n\pi) = 1 \text{ and } \sin(2n\pi) = 0 \text{ for integer } n\}$$
$$= \cos\left(\frac{n\pi x}{L}\right)$$

and

$$\sin\left(\frac{n\pi}{L}(x+2L)\right) = \sin\left(\frac{n\pi x}{L} + 2n\pi\right)$$
$$= \sin\left(\frac{n\pi x}{L}\right)\cos(2n\pi) + \cos\left(\frac{n\pi x}{L}\right)\sin(2n\pi)$$
$$= \sin\left(\frac{n\pi x}{L}\right).$$

Since $\cos\left(\frac{n\pi}{L}(x+2L)\right) = \cos\left(\frac{n\pi x}{L}\right)$ and $\sin\left(\frac{n\pi}{L}(x+2L)\right) = \sin\left(\frac{n\pi x}{L}\right)$, we have $f(x+2L) = f(x)$, which indicates that $f(x)$ is $2L$ periodic. Also notice that $f(x)$ is integrable on the interval $[-L\,,\ L]$ and the proof is complete.

2.7.2 Fourier Series of Even Function

Let $f(x)$ be an even function with a period of $2L$ and integrable on the interval $[-L\,,\ L]$. Then its Fourier series is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \tag{2.84}$$

where

$$a_0 = \frac{1}{L}\int_0^L f(x)\mathrm{d}x,$$
$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x.$$

Proof. Let's recall the definitions of a_0, a_n, and b_n from (2.82):

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\mathrm{d}x$$
$$\{\text{the integrand is an even function}\}$$

$$= \frac{1}{L}\int_0^L f(x)\mathrm{d}x.$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x$$

$$\{\text{the integrand is an even function}\}$$

$$= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x.$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x$$

$$\{\text{the integrand is an odd function}\}$$

$$= 0.$$

The proof follows on plugging a_0, a_n, and b_n in (2.82).

2.7.3 Fourier Series of Odd Function

Let $f(x)$ be an odd function with a period of $2L$ and integrable on the interval $[-L\,,\ L]$. Then its Fourier series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \tag{2.85}$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x.$$

Proof. Let's recall the definitions of a_0, a_n, and b_n from (2.82):

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\mathrm{d}x$$

$$\{\text{the integrand is an odd function}\}$$

$$= 0.$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x$$

$$\{\text{the integrand is an odd function}\}$$

$$= 0.$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x$$

$$\{\text{the integrand is an even function}\}$$
$$= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x.$$

The proof completes on plugging a_0, a_n, and b_n in (2.82).

2.7.4 Fourier Series of $\cos(zx)$

Let $z \notin \mathbb{Z}$. Then the following identity holds:

$$\cos(zx) = \frac{2z\sin(\pi z)}{\pi}\left(\frac{1}{2z^2} - \sum_{n=1}^{\infty}\frac{(-1)^n\cos(nx)}{n^2 - z^2}\right). \tag{2.86}$$

Proof. Since $\cos(zx)$ is an even function, we recall (2.84):

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right),$$

where

$$a_0 = \frac{1}{L}\int_0^L f(x)\mathrm{d}x, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x.$$

Since $\cos(zx) = \cos(zx + 2\pi)$, which indicates that the period of the function is 2π and so $L = \pi$, its Fourier expansion is given by

$$\cos(zx) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx). \tag{2.87}$$

Let's find a_0 and a_n:

$$a_0 = \frac{1}{\pi}\int_0^{\pi}\cos(zx)\mathrm{d}x = \frac{\sin(\pi z)}{\pi z}.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}\cos(zx)\cos(nx)\mathrm{d}x$$
$$\{\text{make use of } 2\cos(x)\cos(y) = \cos(x-y) + \cos(x+y)\}$$
$$= \frac{1}{\pi}\int_0^{\pi}[\cos((z-n)x) + \cos((z+n)x)]\mathrm{d}x$$
$$= \frac{1}{\pi}\left[\frac{\sin((z-n)x)}{z-n} + \frac{\sin((z+n)x)}{z+n}\right]_0^{\pi}$$

$$\begin{aligned}
&= \frac{1}{\pi}\left[\frac{\sin((z-n)\pi)}{z-n} + \frac{\sin((z+n)\pi)}{z+n}\right] \\
&\{\text{use } \sin(x \pm y) = \sin(x) cos(y) \pm \cos(x)\sin(y)\} \\
&= \frac{2}{\pi}\left(\frac{n\cos(\pi z)\sin(\pi n) - z\sin(\pi z)\cos(\pi n)}{n^2 - z^2}\right) \\
&\{\text{note that } \sin(n\pi) = 0 \text{ for integer } n\} \\
&= -\frac{2}{\pi}\left(\frac{z\sin(\pi z)\cos(\pi n)}{n^2 - z^2}\right).
\end{aligned}$$

Substitute the results of a_0 and a_n in (2.87),

$$\begin{aligned}
\cos(zx) &= \frac{2z\sin(\pi z)}{\pi}\left(\frac{1}{2z^2} - \sum_{n=1}^{\infty}\frac{\cos(n\pi)\cos(nx)}{n^2 - z^2}\right) \\
&\left\{\text{use } \sum_{n=1}^{\infty} a_n \cos(n\pi) = \sum_{n=1}^{\infty} (-1)^n a_n \text{ given in (1.7)}\right\} \\
&= \frac{2z\sin(\pi z)}{\pi}\left(\frac{1}{2z^2} - \sum_{n=1}^{\infty}\frac{(-1)^n\cos(nx)}{n^2 - z^2}\right).
\end{aligned}$$

2.7.5 Fourier Series of $\sin(zx)$

Let $z \notin \mathbb{Z}$. Then the following identity holds:

$$\sin(zx) = -\frac{2\sin(\pi z)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n n\sin(nx)}{n^2 - z^2}. \tag{2.88}$$

Proof. Since $\sin(zx)$ is an odd function, we recollect (2.85):

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x.$$

Since $\sin(zx) = \sin(zx + 2\pi)$, which indicates that the function is 2π periodic and so $L = \pi$, its Fourier expansion is given by

$$\sin(zx) = \sum_{n=1}^{\infty} b_n \sin(nx). \tag{2.89}$$

Let's find b_n:

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} \sin(zx)\sin(nx)\mathrm{d}x \\
&\{\text{make use of } 2\sin(x)\sin(y) = \cos(x-y) - \cos(x+y)\} \\
&= \frac{1}{\pi}\int_0^{\pi} [\cos((z-n)x) - \cos((z+n)x)]\mathrm{d}x \\
&= \frac{1}{\pi}\left[\frac{\sin((z-n)x)}{z-n} - \frac{\sin((z+n)x)}{z+n}\right]_0^{\pi} \\
&= \frac{1}{\pi}\left[\frac{\sin((z-n)\pi)}{z-n} - \frac{\sin((z+n)\pi)}{z+n}\right] \\
&\{\text{use } \sin(x \pm y) = \sin(x)cos(y) \pm \cos(x)\sin(y)\} \\
&= \frac{2}{\pi}\left(\frac{z\cos(\pi z)\sin(\pi n) - n\sin(\pi z)\cos(\pi n)}{n^2 - z^2}\right) \\
&\{\text{we have } \sin(n\pi) = 0 \text{ for integer } n\} \\
&= -\frac{2}{\pi}\left(\frac{n\sin(\pi z)\cos(\pi n)}{n^2 - z^2}\right).
\end{aligned}$$

Plugging b_n in (2.89),

$$\begin{aligned}
\sin(zx) &= -\frac{2\sin(\pi z)}{\pi}\sum_{n=1}^{\infty}\frac{\cos(n\pi)n\sin(nx)}{n^2 - z^2} \\
&= -\frac{2\sin(\pi z)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n n\sin(nx)}{n^2 - z^2}.
\end{aligned}$$

2.7.6 Fourier Series of $\ln(\sin x)$

Let $0 < x < \pi$. Then the following identity holds:

$$\ln(\sin x) = -\ln(2) - \sum_{n=1}^{\infty}\frac{\cos(2nx)}{n}. \tag{2.90}$$

Proof (i). Notice that $\ln|\sin x|$ is an even function and has a period of π, since $\ln|\sin x| = \ln|\sin(x+\pi)|$ and so $L = \frac{\pi}{2}$. Thus, based on (2.84), its Fourier expansion is given by

$$\ln|\sin x| = a_0 + \sum_{n=1}^{\infty} a_n\cos(2nx), \tag{2.91}$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \ln|\sin x|\mathrm{d}x, \quad a_n = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \ln|\sin x|\cos(2nx)\mathrm{d}x.$$

We have

$$\begin{aligned}
a_0 &= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \ln|\sin x|\mathrm{d}x \\
&\quad \{\text{note that } \ln|\sin x| = \ln(\sin x) \text{ for } 0 < x < \pi\} \\
&= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \ln(\sin x)\mathrm{d}x \\
&\quad \{\text{this integral is given in (3.89)}\} \\
&= \frac{2}{\pi}\left(-\frac{\pi}{2}\ln(2)\right) = -\ln(2)
\end{aligned}$$

and

$$\begin{aligned}
a_n &= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \ln|\sin x|\cos(2nx)\mathrm{d}x \\
&= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \ln(\sin x)\cos(2nx)\mathrm{d}x \\
&\overset{\text{IBP}}{=} \underbrace{\frac{4}{2\pi n}\sin(2nx)\ln(\sin x)\Big|_0^{\frac{\pi}{2}}}_{0} - \frac{1}{2n}\int_0^{\frac{\pi}{2}} \sin(2nx)\cot(x)\mathrm{d}x \\
&\quad \{\text{this integral is given in (3.88)}\} \\
&= -\frac{1}{2n}\left(\frac{\pi}{2}\right) = -\frac{\pi}{4n}.
\end{aligned}$$

Plug a_0 and a_n in (2.91),

$$\ln|\sin x| = -\ln(2) - \sum_{n=1}^{\infty} \frac{\cos(2nx)}{n}.$$

The proof follows on noticing that $\ln|\sin x| = \ln(\sin x)$ for $0 < x < \pi$.

Proof (ii). By considering the real parts of Euler's formula (1.19), we have

$$\cos(x) = \Re\, e^{ix}.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{\cos(2nx)}{n} = \Re \sum_{n=1}^{\infty} \frac{e^{2inx}}{n} = \Re \sum_{n=1}^{\infty} \frac{(e^{2ix})^n}{n}$$

$$
\begin{aligned}
&= -\Re \ln\left(1 - e^{2ix}\right) \\
&\{\text{write } e^{2ix} = \cos(2x) + i\sin(2x)\} \\
&= -\Re \ln\left(1 - \cos(2x) - i\sin(2x)\right) \\
&\left\{\text{use } \ln(x+iy) = \frac{1}{2}\ln(x^2+y^2) + i\arctan\left(\frac{y}{x}\right), x > 0 \text{ given in (1.18)}\right\} \\
&\{\text{since } 1 - \cos(2x) > 0 \text{ for } 0 < x < \pi\} \\
&= -\Re\left\{\frac{1}{2}\ln[(1-\cos(2x))^2 + \sin^2(2x)] + i\arctan\left(\frac{-\sin(2x)}{1-\cos(2x)}\right)\right\} \\
&\{\text{use } 1 - \cos(2x) = 2\sin^2 x \text{ and } \sin(2x) = 2\sin x\cos x\} \\
&= -\Re\left\{\frac{1}{2}\ln[4\sin^4 x + 4\sin^2 x\cos^2 x] + i\arctan\left(\frac{-2\sin x\cos x}{2\sin^2 x}\right)\right\} \\
&= -\Re\left\{\frac{1}{2}\ln[4\sin^2 x(\sin^2 x + \cos^2 x)] + i\arctan\left(-\cot x\right)\right\} \\
&\left\{\text{write } \arctan(-\cot x) = \arctan\left(-\tan\left(\frac{\pi}{2} - x\right)\right) = -\left(\frac{\pi}{2} - x\right)\right\} \\
&= -\Re\left\{\frac{1}{2}\ln(2\sin x)^2 - i\left(\frac{\pi}{2} - x\right)\right\} \\
&= \Re\left\{-\ln(2\sin x) + i\left(\frac{\pi}{2} - x\right)\right\} \qquad (2.92) \\
&= -\ln(2) - \ln(\sin x).
\end{aligned}
$$

Additionally, by considering the imaginary parts in (2.92), we see that

$$\frac{\pi}{2} - x = \Im\sum_{n=1}^{\infty}\frac{e^{2inx}}{n} = \sum_{n=1}^{\infty}\frac{\sin(2nx)}{n}.$$

Replace x with $\frac{x}{2}$

$$\sum_{n=1}^{\infty}\frac{\sin(nx)}{n} = \frac{\pi}{2} - \frac{x}{2}, \qquad 0 < x < 2\pi.$$

This identity is very useful. To show that, integrate both sides with respect to x,

$$-\sum_{n=1}^{\infty}\frac{\cos(nx)}{n^2} = \frac{\pi}{2}x - \frac{x^2}{4} + c.$$

To find the constant c, set $x = 0$,

$$c = -\sum_{n=1}^{\infty}\frac{1}{n^2} = -\zeta(2) = -\frac{\pi^2}{6}.$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{\cos(nx)}{n^2}=\frac{x^2}{4}-\frac{\pi}{2}x+\frac{\pi^2}{6},\quad 0\le x\le 2\pi.$$

Integrating the latter identity from $x=0$ to x gives

$$\sum_{n=1}^{\infty}\frac{\sin(nx)}{n^3}=\frac{x^3}{12}-\frac{\pi}{4}x^2+\frac{\pi^2}{6}x,\quad 0\le x\le 2\pi$$

and so on.

2.7.7 Fourier Series of $\ln(\cos x)$

Let $|x|<\frac{\pi}{2}$. Then the following identity holds:

$$\ln(\cos x)=-\ln(2)-\sum_{n=1}^{\infty}\frac{(-1)^n\cos(2nx)}{n}. \tag{2.93}$$

Proof (i). Let $x=\frac{\pi}{2}-y$ in (2.90) then write $\sin\left(\frac{\pi}{2}-y\right)=\cos y$, we get

$$\begin{aligned}\ln(\cos y)&=-\ln(2)-\sum_{n=1}^{\infty}\frac{\cos(\pi n-2ny)}{n}\\&\{\text{use }\cos(a-b)=\cos(a)\cos(b)+\sin(a)\sin(b)\}\\&=-\ln(2)-\sum_{n=1}^{\infty}\frac{\cos(\pi n)\cos(2ny)+\sin(\pi n)\sin(2ny)}{n}\\&=-\ln(2)-\sum_{n=1}^{\infty}\frac{(-1)^n\cos(2nx)}{n}.\end{aligned}$$

Proof (ii).

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n\cos(2nx)}{n}&=\Re\sum_{n=1}^{\infty}\frac{(-1)^n e^{2inx}}{n}\\&=\Re\sum_{n=1}^{\infty}\frac{(-e^{2ix})^n}{n}\\&=-\Re\ln\left(1+e^{2ix}\right)\\&=-\Re\ln\left(1+\cos(2x)+i\sin(2x)\right)\\&\left\{\text{use }\ln(x+iy)=\frac{1}{2}\ln(x^2+y^2)+i\arctan\left(\frac{y}{x}\right),x>0\text{ given in (1.18)}\right\}\end{aligned}$$

$$
\begin{aligned}
&\{\text{since } 1+\cos(2x)>0 \text{ for } |x|<\pi/2\}\\
&= -\Re\left\{\frac{1}{2}\ln[(1+\cos(2x))^2+\sin^2(2x)] + i\arctan\left(\frac{\sin(2x)}{1+\cos(2x)}\right)\right\}\\
&\{\text{use } 1+\cos(2x)=2\cos^2 x \text{ and } \sin(2x)=2\sin x\cos x\}\\
&= -\Re\left\{\frac{1}{2}\ln[4\cos^4 x+4\sin^2 x\cos^2 x] + i\arctan\left(\frac{2\sin x\cos x}{2\cos^2 x}\right)\right\}\\
&= -\Re\left\{\frac{1}{2}\ln[4\cos^2 x(\cos^2 x+\sin^2 x)] + i\arctan(\tan x)\right\}\\
&= -\Re\left\{\frac{1}{2}\ln(2\cos x)^2 + ix\right\}\\
&= \Re\{-\ln(2\cos x)-ix\} \qquad (2.94)\\
&= -\ln(2\cos x)\\
&= -\ln(2)-\ln(\cos x).
\end{aligned}
$$

2.7.8 Fourier Series of $\ln(\tan x)$

Let $0<x<\frac{\pi}{2}$. Then the following identity holds:

$$\ln(\tan x) = -2\sum_{n=0}^{\infty}\frac{\cos((4n+2)x)}{2n+1}. \tag{2.95}$$

Proof. Set $a_n=\frac{\cos(2nx)}{n}$ in (1.8):

$$2\sum_{n=0}^{\infty}a_{2n+1}=\sum_{n=1}^{\infty}a_n-\sum_{n=1}^{\infty}(-1)^n a_n,$$

we have

$$
\begin{aligned}
2\sum_{n=0}^{\infty}\frac{\cos(4n+2)x)}{2n+1} &= \sum_{n=1}^{\infty}\frac{\cos(2nx)}{n}-\sum_{n=1}^{\infty}\frac{(-1)^n\cos(2nx)}{n}\\
&\{\text{collect the results (2.90) and (2.93)}\}\\
&= -\ln(2)-\ln(\sin x)-(-\ln(2)-\ln(\cos x))\\
&= -\ln(\sin x)+\ln(\cos x)\\
&= -\ln(\tan x).
\end{aligned}
$$

2.7.9 Series Representation of $\cot(\pi z)$

Let $z \notin \mathbb{Z}$. Then the following identity holds:

$$\pi z \cot(\pi z) = 1 - 2\sum_{n=1}^{\infty} \frac{z^2}{n^2 - z^2}. \tag{2.96}$$

Proof. Set $x = \pi$ in (2.86),

$$\cos(\pi z) = \frac{2z \sin(\pi z)}{\pi}\left(\frac{1}{2z^2} - \sum_{n=1}^{\infty} \frac{(-1)^n \cos(n\pi)}{n^2 - z^2}\right).$$

Make use of $\sum_{n=1}^{\infty} a_n \cos(n\pi) = \sum_{n=1}^{\infty} (-1)^n a_n$ to finish the proof.
If we replace z with iz in (2.96) and use $\cot(iz) = -i\coth(z)$, we obtain

$$\pi z \coth(\pi z) = 1 + 2\sum_{n=1}^{\infty} \frac{z^2}{n^2 + z^2}, \quad z \neq \mathbb{Z}i. \tag{2.97}$$

2.7.10 Series Representation of $\csc(\pi z)$

Let $z \neq \mathbb{Z}$. Then the following identity holds:

$$\pi z \csc(\pi z) = 1 - 2\sum_{n=1}^{\infty} \frac{(-1)^n z^2}{n^2 - z^2}. \tag{2.98}$$

Proof. Set $x = 0$ in (2.86).
A different series representation for this function is

$$\pi z \csc(\pi z) = 2\sum_{k=0}^{\infty} \eta(2k) z^{2k}. \tag{2.99}$$

This can be demonstrated by manipulating the series in (2.98):

$$\begin{aligned}\pi z \csc(\pi z) &= 1 - 2\sum_{n=1}^{\infty} \frac{(-1)^n z^2}{n^2 - z^2} \\ &= 1 - 2\sum_{n=1}^{\infty} (-1)^n \frac{z^2/n^2}{1 - z^2/n^2}\end{aligned}$$

$$
\begin{aligned}
&= 1 - 2\sum_{n=1}^{\infty}(-1)^n\left(\sum_{k=1}^{\infty}\left(z^2/n^2\right)^k\right)\\
&= 1 - 2\sum_{k=1}^{\infty}z^{2k}\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{2k}}\right)\\
&= 1 - 2\sum_{k=1}^{\infty}z^{2k}\left(-\eta(2k)\right)\\
&= 2\left(\frac{1}{2}+\sum_{k=1}^{\infty}\eta(2k)z^{2k}\right)\\
&\{\text{substitute } 1/2 = \eta(0) \text{ given in } (1.54)\}\\
&= 2\sum_{k=0}^{\infty}\eta(2k)z^{2k}.
\end{aligned}
$$

Another form is the Euler's Product Formula of $\csc(\pi z)$:

$$
\pi z\csc(\pi z) = \prod_{n=1}^{\infty}\left(\frac{n^2-z^2}{n^2}\right)^{-1}, \quad z \notin \mathbb{Z}. \tag{2.100}
$$

To prove it, divide both sides of (2.96) by z then integrate from $z = 0$ to x,

$$
\begin{aligned}
\sum_{n=1}^{\infty}\int_0^x \frac{2z}{n^2-z^2}\mathrm{d}z &= \int_0^x\left(\frac{1}{z}-\pi\cot(\pi z)\right)\mathrm{d}z\\
&= \ln(z) + \ln\csc(\pi z)\Big|_0^x\\
&= \ln\left(x\csc(\pi x)\right) - \lim_{z\to 0}\ln\left(z\csc(\pi z)\right)\\
&\{\text{use } \lim_{z\to 0} z\csc(\pi z) = 1/\pi\}\\
&= \ln\left(x\csc(\pi x)\right) + \ln(\pi)\\
&= \ln\left(\pi x\csc(\pi x)\right).
\end{aligned}
$$

For the sum,

$$
\begin{aligned}
\sum_{n=1}^{\infty}\int_0^x \frac{2z}{n^2-z^2}\mathrm{d}z &= \sum_{n=1}^{\infty} -\ln(n^2-z^2)\Big|_0^x\\
&= \sum_{n=1}^{\infty}\ln\left(\frac{n^2-x^2}{n^2}\right)^{-1}\\
&\{\text{make use of the rule } (1.15)\}
\end{aligned}
$$

$$= \ln \prod_{n=1}^{\infty} \left(\frac{n^2 - x^2}{n^2} \right)^{-1}.$$

Thus, we have

$$\ln \prod_{n=1}^{\infty} \left(\frac{n^2 - x^2}{n^2} \right)^{-1} = \ln \left(\pi x \csc(\pi x) \right).$$

Exponentiating both sides with base e completes the proof.

Replace z with iz in (2.98) and (2.100) then use $\csc(iz) = -i \operatorname{csch}(z)$,

$$\pi z \operatorname{csch}(\pi z) = 1 + 2 \sum_{n=1}^{\infty} \frac{(-1)^n z^2}{n^2 + z^2}, \quad z \neq \mathbb{Z}i \tag{2.101}$$

$$\pi z \operatorname{csch}(\pi z) = -\prod_{n=1}^{\infty} \left(\frac{n^2 + z^2}{n^2} \right)^{-1}, \quad z \neq \mathbb{Z}i. \tag{2.102}$$

2.7.11 Series Representation of sec $\left(\frac{\pi}{2} z\right)$

Let $z \neq 2\mathbb{Z} + 1$. Then the following identity holds:

$$\sec\left(\frac{\pi}{2} z\right) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{(2n+1)^2 - z^2}. \tag{2.103}$$

Proof. Set $x = \frac{\pi}{2}$ in (2.88), we obtain

$$\frac{\sin\left(\frac{\pi}{2} z\right)}{\sin(\pi z)} = -\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n n \sin\left(\frac{\pi}{2} n\right)}{n^2 - z^2}$$

$$\left\{ \text{use } \sum_{n=1}^{\infty} a_n \sin\left(\frac{\pi}{2} n\right) = \sum_{n=0}^{\infty} (-1)^n a_{2n+1} \text{ given in (1.14)} \right\}$$

$$= \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{(2n+1)^2 - z^2}.$$

The proof completes on writing

$$\frac{\sin\left(\frac{\pi}{2} z\right)}{\sin(\pi z)} = \frac{\sin\left(\frac{\pi}{2} z\right)}{2 \sin\left(\frac{\pi}{2} z\right) \cos\left(\frac{\pi}{2} z\right)} = \frac{1}{2} \sec\left(\frac{\pi}{2} z\right).$$

Replace z with iz then use $\sec(iz) = \operatorname{sech}(z)$,

$$\operatorname{sech}\left(\frac{\pi}{2}z\right) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)}{(2n+1)^2+z^2}, \quad z \neq (2\mathbb{Z}+1)i. \tag{2.104}$$

2.7.12 Series Representation of $\sin(x)$

Let $|x| < \frac{\pi}{2}$. Then the following identity holds:

$$\sin(x) = -\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n n\sin(2nx)}{4n^2-1}. \tag{2.105}$$

Proof. The proof follows from (2.88) on setting $z = \frac{1}{2}$ then replacing x with $2x$.

2.7.13 Series Representation of $\tan x \ln(\sin x)$

Let $0 < x < \pi$. Then the following identity holds:

$$\tan x \ln(\sin x) = -\sum_{n=1}^{\infty}\left(\int_0^1 \frac{1-t}{1+t}t^{n-1}\,\mathrm{d}t\right)\sin(2nx). \tag{2.106}$$

The following proof may be found in [32, p. 243]:
Proof. By the change of variable $t = e^{-2y}$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left(\int_0^1 \frac{1-t}{1+t}t^{n-1}\mathrm{d}t\right)\sin(2nx)\\
&= 2\sum_{n=1}^{\infty}\left(\int_0^{\infty}\frac{1-e^{-2y}}{1+e^{-2y}}e^{-2ny}\mathrm{d}y\right)\sin(2nx)\\
&= 2\sum_{n=1}^{\infty}\left(\int_0^{\infty}\tanh(y)e^{-2ny}\mathrm{d}y\right)\sin(2nx)\\
&= 2\int_0^{\infty}\tanh(y)\left(\sum_{n=1}^{\infty}\sin(2nx)e^{-2ny}\right)\mathrm{d}y\\
&= 2\int_0^{\infty}\tanh(y)\left(\Im\sum_{n=1}^{\infty}e^{2inx}e^{-2ny}\right)\mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&= 2\int_0^\infty \tanh(y)\left(\Im\sum_{n=1}^\infty \left(e^{2ix-2y}\right)^n\right)\mathrm{d}y\\
&= 2\int_0^\infty \tanh(y)\left(\Im\,\frac{e^{2ix-2y}}{1-e^{2ix-2y}}\right)\mathrm{d}y\\
&= 2\int_0^\infty \tanh(y)\left(\frac{1}{2}\frac{\sin(2x)}{\cosh(2y)-\cos(2x)}\right)\mathrm{d}y\\
&= \sin(2x)\int_0^\infty \frac{\sinh(y)}{\cosh(y)(2\cosh^2(y)-1-\cos(2x))}\mathrm{d}y\\
&\qquad\{\text{set } \cosh(y)=1/t\}\\
&= \sin(2x)\int_0^1 \frac{t}{2-(1+\cos(2x))t^2}\mathrm{d}t\\
&\qquad\{\text{use } 1+\cos(2x)=2\cos^2(x)\}\\
&= \frac{1}{2}\sin(2x)\int_0^1 \frac{t}{1-\cos^2(x)t^2}\mathrm{d}t\\
&= \frac{1}{2}\sin(2x)\left[-\frac{\ln(1-\cos^2(x)t^2)}{2\cos^2 x}\right]_{t=0}^{t=1}\\
&= -\frac{1}{2}\cdot 2\sin x\cos x\left(\frac{\ln(\sin x)}{\cos^2 x}\right)\\
&= -\tan x\ln(\sin x).
\end{aligned}
$$

Note that the value $x = \frac{\pi}{2}$ is included in the domain of $\tan x \ln(\sin x)$, since $\lim\limits_{x\to 0} \tan x\ln(\sin x) = 0$ by L'Hôpital's rule.

Further, by letting $x \to \frac{\pi}{2} - x$ in (2.106) and using $\sin(\frac{\pi}{2} - x) = \cos(x)$, $\tan(\frac{\pi}{2} - x) = \cot(x)$, and $\sin(2n(\frac{\pi}{2} - x)) = -(-1)^n \sin(2nx)$ for integer n, we get

$$
\cot x\ln(\cos x) = \sum_{n=1}^\infty (-1)^n\left(\int_0^1 \frac{1-t}{1+t}t^{n-1}\mathrm{d}t\right)\sin(2nx), \quad |x| < \frac{\pi}{2}. \tag{2.107}
$$

Keep in mind that $\lim\limits_{x\to 0} \cot x\ln(\cos x) = 0$ justifies why the value $x = 0$ is included in the domain of the function. Additionally, by using the identity

$$
\begin{aligned}
\tan x\ln(\sin x) + \cot x\ln(\cos x) &= \frac{\sin x}{\cos x}\ln(\sin x) + \frac{\cos x}{\sin x}\ln(\cos x)\\
&= \frac{\sin^2 x\ln(\sin x) + \cos^2 x\ln(\cos x)}{\sin x\cos x}\\
&= \frac{\sin^2 x\ln(\sin x) + (1-\sin^2 x)\ln(\cos x)}{\sin x\cos x}\\
&= \frac{\sin^2 x(\ln(\sin x) - \ln(\cos x)) + \ln(\cos x)}{\sin x\cos x}
\end{aligned}
$$

$$= \tan x \ln(\tan x) + 2\csc(2x)\ln(\cos x),$$

we find, for $0 < x < \frac{\pi}{2}$, that

$$\begin{aligned}
&\tan x \ln(\tan x) + 2\csc(2x)\ln(\cos x) \\
&= -\sum_{n=1}^{\infty}(1-(-1)^n)\left(\int_0^1 \frac{1-t}{1+t}t^{n-1}\mathrm{d}t\right)\sin(2nx) \\
&\qquad \{\text{make use of (1.8)}\} \\
&= -2\sum_{n=0}^{\infty}\left(\int_0^1 \frac{1-t}{1+t}t^{2n}\mathrm{d}t\right)\sin((4n+2)x). \qquad (2.108)
\end{aligned}$$

2.7.14 Series Representation of $\ln^2(2\cos x)$

Let $|x| < \frac{\pi}{2}$. Then the following identity holds:

$$\ln^2(2\cos x) = x^2 + 2\sum_{n=1}^{\infty}(-1)^n \frac{H_{n-1}}{n}\cos(2nx). \qquad (2.109)$$

Proof. Since $\cos x = \Re\, e^{ix}$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty}(-1)^n \frac{H_{n-1}}{n}\cos(2nx) &= \Re\sum_{n=1}^{\infty}(-1)^n \frac{H_{n-1}}{n}e^{2inx} \\
&= \Re\sum_{n=1}^{\infty}\frac{H_{n-1}}{n}\left(-e^{2ix}\right)^n \\
&\qquad \{\text{replace } x \text{ with } -e^{2ix} \text{ in the generating function (2.7)}\} \\
&= \frac{1}{2}\Re\ln^2(1+e^{2ix}) \\
&\qquad \{\text{recall the result of } \ln(1+e^{2ix}) \text{ from (2.94)}\} \\
&= \frac{1}{2}\Re\left(\ln(2\cos x) + ix\right)^2 \\
&= \frac{1}{2}\Re\left(\ln^2(2\cos x) + 2ix\ln(2\cos x) - x^2\right) \\
&= \frac{1}{2}\ln^2(2\cos x) - \frac{x^2}{2}.
\end{aligned}$$

If we replace x with $\frac{\pi}{2}-x$ in (2.109) using $\cos(\frac{\pi}{2}-x)=\sin(x)$ and $(\cos(2n(\frac{\pi}{2}-x)))=(-1)^n\cos(2nx)$ for integer n, we have

$$\ln^2(2\sin x)=\left(\frac{\pi}{2}-x\right)^2+2\sum_{n=1}^{\infty}\frac{H_{n-1}}{n}\cos(2nx),\quad 0<x<\pi. \tag{2.110}$$

2.7.15 Series Representation of $\ln^3(2\cos x)$

Let $|x|<\frac{\pi}{2}$. Then the following identity holds:

$$\ln^3(2\cos x)=3x^2\ln(2\cos x)-3\sum_{n=1}^{\infty}(-1)^n\frac{H_{n-1}^2-H_{n-1}^{(2)}}{n}\cos(2nx). \tag{2.111}$$

Proof.

$$\begin{aligned}\sum_{n=1}^{\infty}\left(H_{n-1}^2-H_{n-1}^{(2)}\right)\frac{x^n}{n}&=\sum_{n=1}^{\infty}\left(H_{n-1}^2-H_{n-1}^{(2)}\right)\int_0^x x^{n-1}\mathrm{d}x\\&\{\text{let the index start from 2 then shift it by }+1\}\\&=\int_0^x\left(\sum_{n=1}^{\infty}(H_n^2-H_n^{(2)})x^n\right)\mathrm{d}x\\&\{\text{recall the generating function (2.60)}\}\\&=\int_0^x\left(\frac{\ln^2(1-x)}{1-x}\right)\mathrm{d}x\\&=-\frac{1}{3}\ln^3(1-x).\end{aligned}$$

Replace x with $-e^{2ix}$ then consider the real parts using $\Re\left\{e^{2inx}\right\}=\cos(2nx)$,

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^n\frac{H_{n-1}^2-H_{n-1}^{(2)}}{n}\cos(2nx)&=-\frac{1}{3}\Re\ln^3(1+e^{2ix})\\&=-\frac{1}{3}\Re\left(\ln(2\cos x)+ix\right)^3\\&=-\frac{1}{3}\left(-3x^2\ln(2\cos x)+\ln^3(2\cos x)\right).\end{aligned}$$

If we replace x with $\frac{\pi}{2} - x$ in (2.109), we obtain

$$\ln^3(2\sin x) = 3\left(\frac{\pi}{2} - x\right)^2 \ln(2\sin x) - 3\sum_{n=1}^{\infty} \frac{H_{n-1}^2 - H_{n-1}^{(2)}}{n} \cos(2nx). \quad (2.112)$$

2.7.16 Series Representation of $\ln^a(2\cos x)$

Let $|x| < \frac{\pi}{2}$ and $a \in \mathbb{Z}_{>1}$, Then the following identity holds:

$$\begin{aligned} &\sum_{k=0}^{\lfloor \frac{a}{2} \rfloor} \binom{a}{2k} \left(-x^2\right)^k \ln^{a-2k}(2\cos x) \\ &= (-1)^a a \sum_{n=1}^{\infty} \frac{(-1)^n}{n} f(a-1, n-1) \cos(2nx), \end{aligned} \quad (2.113)$$

where

$$f(a,n) = (-1)^{a-1}(a-1)! \left(H_n^{(a)} + \sum_{j=1}^{a-1} \frac{(-1)^j}{j!} f(j,n) H_n^{(a-j)} \right).$$

Proof. Applying the previous proof,

$$\begin{aligned} \sum_{n=1}^{\infty} f(a-1, n-1) \frac{x^n}{n} &= \sum_{n=1}^{\infty} f(a-1, n-1) \int_0^x x^{n-1} \mathrm{d}x \\ &\quad \{\text{let the index start from 2 then shift it by } +1\} \\ &= \int_0^x \left(\sum_{n=1}^{\infty} f(a-1, n) x^n \right) \mathrm{d}x \\ &\quad \{\text{recall the generating function (2.60)}\} \\ &= \int_0^x \left((-1)^{a-1} \frac{\ln^{a-1}(1-x)}{1-x} \right) \mathrm{d}x \\ &= \frac{(-1)^a}{a} \ln^a(1-x). \end{aligned}$$

Replace x with $-e^{2ix}$ then consider the real parts,

$$\begin{aligned} (-1)^a a \sum_{n=1}^{\infty} \frac{f(a-1, n-1)(-1)^n}{n} \cos(2nx) &= \Re\left\{\ln^a\left(1 + e^{2ix}\right)\right\} \\ &= \Re\left(\ln(2\cos x) + ix\right)^a \end{aligned}$$

$$\left\{\text{use the binomial theorem } (x+y)^a = \sum_{k=0}^{a} \binom{a}{k} y^{a-k}\, x^k\right\}$$

$$= \Re \sum_{k=0}^{a} \binom{a}{k} (ix)^k \ln^{a-k}(2\cos x)$$

$$\left\{\text{use } \sum_{k=0}^{a} f(k) = \sum_{k=0}^{\lfloor \frac{a}{2} \rfloor} f(2k) + \sum_{k=0}^{\lfloor \frac{a-1}{2} \rfloor} f(2k+1)\right\}$$

$$= \Re \left\{ \sum_{k=0}^{\lfloor \frac{a}{2} \rfloor} \binom{a}{2k} (-1)^k x^{2k} \ln^{a-2k}(2\cos x) \right.$$

$$\left. + \, i \sum_{k=0}^{\lfloor \frac{a-1}{2} \rfloor} \binom{a}{2k+1} (-1)^k x^{2k+1} \ln^{a-2k-1}(2\cos x) \right\}$$

{ignore the second sum, since we are after the real part}

$$= \sum_{k=0}^{\lfloor \frac{a}{2} \rfloor} \binom{a}{2k} \left(-x^2\right)^k \ln^{a-2k}(2\cos x).$$

Examples:

$$\ln^2(2\cos x) - x^2 = 2 \sum_{n=1}^{\infty} (-1)^n \frac{H_{n-1}}{n} \cos(2nx);$$

$$\ln^3(2\cos x) - 3x^2 \ln(2\cos x) = -3 \sum_{n=1}^{\infty} (-1)^n \frac{H_{n-1}^2 - H_{n-1}^{(2)}}{n} \cos(2nx);$$

$$\ln^4(2\cos x) - 6x^2 \ln^2(2\cos x) + x^4$$

$$= 4 \sum_{n=1}^{\infty} (-1)^n \frac{H_{n-1}^3 - 3H_{n-1}H_{n-1}^{(2)} + 2H_{n-1}^{(3)}}{n} \cos(2nx);$$

$$\ln^5(2\cos x) - 10x^2 \ln^3(2\cos x) + 5x^4 \ln(2\cos x)$$

$$= \sum_{n=1}^{\infty} \frac{H_{n-1}^4 - 6H_{n-1}^2 H_{n-1}^{(2)} + 8H_{n-1}H_{n-1}^{(3)} + 3\left(H_{n-1}^{(2)}\right)^2 - 6H_{n-1}^{(4)}}{(-1)^n n} \cos(2nx).$$

Let's integrate both sides of (2.113) from $x = 0 \to \frac{\pi}{2}$ to get

$$\int_0^{\frac{\pi}{2}} \left(\sum_{k=0}^{\lfloor \frac{a}{2} \rfloor} \binom{a}{2k} \left(-x^2\right)^k \ln^{a-2k}(2\cos x) \right) \mathrm{d}x$$

$$
\begin{aligned}
&= (-1)^a a \sum_{n=1}^{\infty} \frac{(-1)^n}{n} f(a-1,n-1) \int_0^{\frac{\pi}{2}} \cos(2nx)\mathrm{d}x \\
&= (-1)^a a \sum_{n=1}^{\infty} \frac{(-1)^n}{n} f(a-1,n-1) \frac{\sin(n\pi)}{2n}.
\end{aligned}
$$

Since $\sin(n\pi) = 0$ for integer n, the integral simplifies to

$$
\int_0^{\frac{\pi}{2}} \left(\sum_{k=0}^{\lfloor \frac{a}{2} \rfloor} \binom{a}{2k} \left(-x^2\right)^k \ln^{a-2k}(2\cos x) \right) \mathrm{d}x = 0. \tag{2.114}
$$

Replacing x with $\frac{\pi}{2} - x$ in (2.109) and (2.114) gives

$$
\begin{aligned}
&\sum_{k=0}^{\lfloor \frac{a}{2} \rfloor} \binom{a}{2k} (-1)^k \left(\frac{\pi}{2} - x\right)^{2k} \ln^{a-2k}(2\sin x) \\
&\quad = (-1)^a a \sum_{n=1}^{\infty} \frac{f(a-1,n-1)}{n} \cos(2nx)
\end{aligned}
$$

and

$$
\int_0^{\frac{\pi}{2}} \left(\sum_{k=0}^{\lfloor \frac{a}{2} \rfloor} \binom{a}{2k} (-1)^k \left(\frac{\pi}{2} - x\right)^{2k} \ln^{a-2k}(2\sin x) \right) \mathrm{d}x = 0.
$$

Chapter 3

Logarithmic Integrals

3.1 Generalized Logarithmic Integrals

3.1.1 $\int_0^1 \frac{x^{2s}}{1+x}\mathrm{d}x$

Let $\Re(s) > -\frac{1}{2}$. Then the following identity holds:

$$\int_0^1 \frac{x^{2s}}{1+x}\mathrm{d}x = \ln(2) + H_s - H_{2s}. \tag{3.1}$$

Proof.

$$\begin{aligned}
\int_0^1 \frac{x^{2s}}{1+x}\mathrm{d}x &= \int_0^1 \frac{1-1+x^{2s}}{1+x}\mathrm{d}x \\
&= \int_0^1 \frac{\mathrm{d}x}{1+x} - \int_0^1 \frac{1-x^{2s}}{1+x}\mathrm{d}x \\
&\left\{\text{write } \frac{1}{1+x} = \frac{1}{1-x} - \frac{2x}{1-x^2}\right\} \\
&= \ln(2) - \int_0^1 \frac{1-x^{2s}}{1-x}\mathrm{d}x + \underbrace{\int_0^1 \frac{2x(1-x^{2s})}{1-x^2}\mathrm{d}x}_{x=\sqrt{y}} \\
&= \ln(2) - \int_0^1 \frac{1-x^{2s}}{1-x}\mathrm{d}x + \int_0^1 \frac{1-y^s}{1-y}\mathrm{d}y \\
&\{\text{recall the integral form of the harmonic number (1.123)}\} \\
&= \ln(2) - H_{2s} + H_s.
\end{aligned}$$

Moreover, forcing integration by part yields

$$\int_0^1 \frac{x^{2s}}{1+x}\mathrm{d}x = \ln(2) - 2s\int_0^1 x^{2s-1}\ln(1+x)\mathrm{d}x$$

or

$$\int_0^1 x^{2s-1}\ln(1+x)\mathrm{d}x = \frac{H_{2s}-H_s}{2s}. \tag{3.2}$$

An alternative proof for (3.2) may be found in [28].

Let's replace $2s$ with s in (3.1),

$$\int_0^1 \frac{x^s}{1+x}\mathrm{d}x = \ln(2) + H_{\frac{s}{2}} - H_s. \tag{3.3}$$

This integral can be expressed in a different closed form:

$$\int_0^1 \frac{x^s}{1+x}\mathrm{d}x = \frac{1}{2}\left(H_{\frac{s}{2}} - H_{\frac{s-1}{2}}\right), \quad \Re(s) > -1. \tag{3.4}$$

To show that, use the integral form of the harmonic number,

$$\begin{aligned}
H_{\frac{s}{2}} - H_{\frac{s-1}{2}} &= \int_0^1 \frac{1-x^{\frac{s}{2}}}{1-x}\mathrm{d}x - \int_0^1 \frac{1-x^{\frac{s-1}{2}}}{1-x}\mathrm{d}x \\
&= \int_0^1 \frac{x^{\frac{s-1}{2}} - x^{\frac{s}{2}}}{1-x}\mathrm{d}x \\
&\overset{x=y^2}{=} 2\int_0^1 \frac{y^s - y^{s+1}}{1-y^2}\mathrm{d}y \\
&= 2\int_0^1 \frac{y^s(1-y)}{1-y^2}\mathrm{d}y \\
&= 2\int_0^1 \frac{y^s}{1+y}\mathrm{d}y.
\end{aligned}$$

Employing $H_s = \psi(s+1) + \gamma$ given in (1.146) yields

$$\begin{aligned}
H_{\frac{s}{2}} - H_{\frac{s-1}{2}} &= \psi\left(\frac{s+2}{2}\right) + \gamma - \left(\psi\left(\frac{s+1}{2}\right) + \gamma\right) \\
&= \psi\left(\frac{s+2}{2}\right) - \psi\left(\frac{s+1}{2}\right).
\end{aligned}$$

Consequently,

$$\int_0^1 \frac{x^s}{1+x}\mathrm{d}x = \frac{1}{2}\psi\left(\frac{s+2}{2}\right) - \frac{1}{2}\psi\left(\frac{s+1}{2}\right). \tag{3.5}$$

By comparing (3.1) and (3.4), we deduce that

$$H_{s-\frac{1}{2}} = 2H_{2s} - H_s - 2\ln(2). \tag{3.6}$$

3.1.2 $\int_0^1 x^{2s-1} \operatorname{arctanh}(x) \mathrm{d}x$

Let $\Re(s) > -\frac{1}{2}$. Then the following identity holds:

$$\int_0^1 x^{2s-1} \operatorname{arctanh}(x) \mathrm{d}x = \frac{2H_{2s} - H_s}{4s}. \tag{3.7}$$

Proof. By writing $\operatorname{arctanh}(x) = \frac{1}{2}\ln(1+x) - \frac{1}{2}\ln(1-x)$, we have

$$\begin{aligned} &\int_0^1 x^{2s-1} \operatorname{arctanh}(x) \mathrm{d}x \\ &= \frac{1}{2}\int_0^1 x^{2s-1}\ln(1+x)\mathrm{d}x - \frac{1}{2}\int_0^1 x^{2s-1}\ln(1-x)\mathrm{d}x \\ &\{\text{collect the results (3.2) and (1.124)}\} \\ &= \frac{1}{2}\cdot\frac{H_{2s} - H_s}{2s} + \frac{1}{2}\cdot\frac{H_{2s}}{2s} \\ &= \frac{2H_{2s} - H_s}{4s}. \end{aligned}$$

3.1.3 $\int_0^1 \frac{\ln^q(x)}{1-x} \mathrm{d}x$

Let $q \in \mathbb{Z}^+$. Then the following identity holds:

$$\int_0^1 \frac{\ln^q(x)}{1-x}\mathrm{d}x = (-1)^q q!\,\zeta(q+1). \tag{3.8}$$

Proof. Expand $\frac{1}{1-x}$ in series then interchange integration and summation,

$$\begin{aligned} \int_0^1 \frac{\ln^q(x)}{1-x}\mathrm{d}x &= \sum_{n=1}^{\infty}\int_0^1 x^{n-1}\ln^q(x)\mathrm{d}x \\ &\{\text{make use of (1.36)}\} \\ &= (-1)^q q! \sum_{n=1}^{\infty}\frac{1}{n^{q+1}} \end{aligned}$$

$$= (-1)^q q!\,\zeta(q+1).$$

Examples:

$$\int_0^1 \frac{\ln(x)}{1-x}\mathrm{d}x = -\zeta(2);$$
$$\int_0^1 \frac{\ln^2(x)}{1-x}\mathrm{d}x = 2\zeta(3);$$
$$\int_0^1 \frac{\ln^3(x)}{1-x}\mathrm{d}x = -6\zeta(4);$$
$$\int_0^1 \frac{\ln^4(x)}{1-x}\mathrm{d}x = 24\zeta(5).$$

The following two integrals are related:

$$\begin{aligned}\int_0^1 \frac{\ln^{q-1}(x)\ln(1-x)}{x}\mathrm{d}x &\overset{\text{IBP}}{=} \frac{1}{q}\int_0^1 \frac{\ln^q(x)}{1-x}\mathrm{d}x\\ &= (-1)^q(q-1)!\zeta(q+1). \qquad (3.9)\end{aligned}$$

$$\begin{aligned}\int_0^1 \frac{x\ln^q(x)}{1-x^2}\mathrm{d}x &\overset{x=\sqrt{y}}{=} \frac{1}{2^{q+1}}\int_0^1 \frac{\ln^q(y)}{1-y}\mathrm{d}y\\ &= \frac{(-1)^q q!}{2^{q+1}}\zeta(q+1). \qquad (3.10)\end{aligned}$$

3.1.4 $\int_0^1 \frac{\ln^q(x)}{1+x}\mathrm{d}x$

Let $q \in \mathbb{Z}^+$. Then the following identity holds:

$$\int_0^1 \frac{\ln^q(x)}{1+x}\mathrm{d}x = (-1)^q q!\left(1-\frac{1}{2^q}\right)\zeta(q+1). \qquad (3.11)$$

Proof. By writing

$$\frac{1}{1+x} = \frac{1}{1-x} - \frac{2x}{1-x^2},$$

we get

$$\int_0^1 \frac{\ln^q(x)}{1+x}\mathrm{d}x = \int_0^1 \frac{\ln^q(x)}{1-x}\mathrm{d}x - \underbrace{\int_0^1 \frac{2x\ln^q(x)}{1-x^2}\mathrm{d}x}_{x^2\to x}$$

$$= \int_0^1 \frac{\ln^q(x)}{1-x}\mathrm{d}x - \frac{1}{2^q}\int_0^1 \frac{\ln^q(x)}{1-x}\mathrm{d}x$$
$$= \left(1-\frac{1}{2^q}\right)\int_0^1 \frac{\ln^q(x)}{1-x}\mathrm{d}x.$$

This integral is given in (3.8).

Examples:

$$\int_0^1 \frac{\ln(x)}{1+x}\mathrm{d}x = -\frac{1}{2}\zeta(2);$$
$$\int_0^1 \frac{\ln^2(x)}{1+x}\mathrm{d}x = \frac{3}{2}\zeta(3);$$
$$\int_0^1 \frac{\ln^3(x)}{1+x}\mathrm{d}x = -\frac{21}{4}\zeta(4);$$
$$\int_0^1 \frac{\ln^4(x)}{1+x}\mathrm{d}x = \frac{45}{2}\zeta(5).$$

The following two integrals are related:

$$\int_0^1 \frac{\ln^{q-1}(x)\ln(1+x)}{x}\mathrm{d}x \overset{\text{IBP}}{=} \frac{-1}{q}\int_0^1 \frac{\ln^q(x)}{1+x}\mathrm{d}x$$
$$= (-1)^{q-1}(q-1)!\left(1-\frac{1}{2^q}\right)\zeta(q+1). \tag{3.12}$$

$$\int_0^1 \frac{x\ln^q(x)}{1+x^2}\mathrm{d}x \overset{x=\sqrt{y}}{=} \frac{1}{2^{q+1}}\int_0^1 \frac{\ln^q(y)}{1+y}\mathrm{d}y$$
$$= \frac{(-1)^q q!}{2^{q+1}}\left(1-\frac{1}{2^q}\right)\zeta(q+1). \tag{3.13}$$

3.1.5 $\int_0^1 \frac{\ln^q(x)}{1-x^2}\mathrm{d}x$

Let $q \in \mathbb{Z}^+$. Then the following identity holds:

$$\int_0^1 \frac{\ln^q(x)}{1-x^2}\mathrm{d}x = (-1)^q q!\left(1-\frac{1}{2^{q+1}}\right)\zeta(q+1). \tag{3.14}$$

Proof. By writing

$$\frac{1}{1-x^2} = \frac{1}{1-x} - \frac{x}{1-x^2},$$

we get

$$\begin{aligned}\int_0^1 \frac{\ln^q(x)}{1-x^2}\mathrm{d}x &= \int_0^1 \frac{\ln^q(x)}{1-x}\mathrm{d}x - \underbrace{\int_0^1 \frac{x\ln^q(x)}{1-x^2}\mathrm{d}x}_{x\to\sqrt{x}} \\ &= \int_0^1 \frac{\ln^q(x)}{1-x}\mathrm{d}x - \frac{1}{2^{q+1}}\int_0^1 \frac{\ln^q(x)}{1-x}\mathrm{d}x \\ &= \left(1-\frac{1}{2^{q+1}}\right)\int_0^1 \frac{\ln^q(x)}{1-x}\mathrm{d}x \\ &= (-1)^q q!\left(1-\frac{1}{2^{q+1}}\right)\zeta(q+1).\end{aligned}$$

Examples:

$$\begin{aligned}\int_0^1 \frac{\ln(x)}{1-x^2}\mathrm{d}x &= -\frac{3}{4}\zeta(2); \\ \int_0^1 \frac{\ln^2(x)}{1-x^2}\mathrm{d}x &= \frac{7}{4}\zeta(3); \\ \int_0^1 \frac{\ln^3(x)}{1-x^2}\mathrm{d}x &= -\frac{45}{8}\zeta(4); \\ \int_0^1 \frac{\ln^4(x)}{1-x^2}\mathrm{d}x &= \frac{93}{4}\zeta(5).\end{aligned}$$

The following two integrals are related:

By making the substitution $\frac{1-x}{1+x} = y$, we get

$$\begin{aligned}\int_0^1 \frac{\ln^q\left(\frac{1-x}{1+x}\right)}{x}\mathrm{d}x &= 2\int_0^1 \frac{\ln^q(y)}{1-y^2}\mathrm{d}y \\ &= 2(-1)^q q!\left(1-\frac{1}{2^{q+1}}\right)\zeta(q+1). \qquad (3.15)\end{aligned}$$

By integration by parts, we have

$$\begin{aligned}\int_0^1 \frac{\ln\left(\frac{1-x}{1+x}\right)\ln^{q-1}(x)}{x}\mathrm{d}x &= \frac{2}{q}\int_0^1 \frac{\ln^q(x)}{1-x^2}\mathrm{d}x \\ &= 2(-1)^q (q-1)!\left(1-\frac{1}{2^{q+1}}\right)\zeta(q+1). \qquad (3.16)\end{aligned}$$

3.1.6 $\int_0^1 \frac{\ln^q(1-x)}{1+x}\mathrm{d}x$

Let $q \in \mathbb{Z}^+$. Then the following identity holds:

$$\int_0^1 \frac{\ln^q(1-x)}{1+x}\mathrm{d}x = (-1)^q q! \operatorname{Li}_{q+1}\left(\frac{1}{2}\right). \tag{3.17}$$

Proof. We begin with the change of variable $1 - x = t$,

$$\begin{aligned}
\int_0^1 \frac{\ln^q(1-x)}{1+x}\mathrm{d}x &= \int_0^1 \frac{\ln^q(t)}{2-t}\mathrm{d}t \\
&\left\{\text{expand } \frac{1}{2-t} \text{ in series as } \sum_{n=1}^{\infty} \frac{t^{n-1}}{2^n}\right\} \\
&= \sum_{n=1}^{\infty} \frac{1}{2^n} \int_0^1 t^{n-1} \ln^q(t)\mathrm{d}t \\
&\{\text{make use of } (1.36)\} \\
&= (-1)^q q! \sum_{n=1}^{\infty} \frac{1}{2^n n^{q+1}} \\
&= (-1)^q q! \operatorname{Li}_{q+1}\left(\frac{1}{2}\right).
\end{aligned}$$

Examples:

$$\int_0^1 \frac{\ln(1-x)}{1+x}\mathrm{d}x = -\operatorname{Li}_2\left(\frac{1}{2}\right) = -\frac{1}{2}\zeta(2) + \frac{1}{2}\ln^2(2);$$

$$\int_0^1 \frac{\ln^2(1-x)}{1+x}\mathrm{d}x = 2\operatorname{Li}_3\left(\frac{1}{2}\right) = \frac{7}{4}\zeta(3) - \ln(2)\zeta(2) + \frac{1}{3}\ln^3(2);$$

$$\int_0^1 \frac{\ln^3(1-x)}{1+x}\mathrm{d}x = -6\operatorname{Li}_4\left(\frac{1}{2}\right);$$

$$\int_0^1 \frac{\ln^4(1-x)}{1+x}\mathrm{d}x = 24\operatorname{Li}_5\left(\frac{1}{2}\right).$$

The values of $\operatorname{Li}_2\left(\frac{1}{2}\right)$ and $\operatorname{Li}_3\left(\frac{1}{2}\right)$ are given in (1.81) and (1.92).

3.1.7 $\int_0^{\frac{1}{2}} \frac{z \ln^q(x)}{1-zx} \mathrm{d}x$

Let $q \in \mathbb{Z}^+$ and $z \in \mathbb{C} \setminus [2, \infty)$. Then the following identity holds:

$$\int_0^{\frac{1}{2}} \frac{z \ln^q(x)}{1-zx} \mathrm{d}x = (-1)^q q! \sum_{k=0}^{q} \frac{\ln^{q-k}(2)}{(q-k)!} \operatorname{Li}_{k+1}\left(\frac{z}{2}\right). \tag{3.18}$$

Proof. Let $y = 2x$,

$$\begin{aligned}
\int_0^{\frac{1}{2}} \frac{z \ln^q(x)}{1-zx} \mathrm{d}x &= \int_0^1 \frac{\frac{z}{2}[\ln(y)-\ln(2)]^q}{1-\frac{z}{2}y} \mathrm{d}y \\
&\left\{\text{use the binomial theorem } (x-y)^q = \sum_{k=0}^{q} \binom{q}{k} (-y)^{q-k} x^k \right\} \\
&= \sum_{k=0}^{q} \binom{q}{k} (-\ln(2))^{q-k} \int_0^1 \frac{\frac{z}{2} \ln^k(y)}{1-\frac{z}{2}y} \mathrm{d}y \\
&\{\text{use the definition } (1.75)\} \\
&= \sum_{k=0}^{q} \binom{q}{k} (-\ln(2))^{q-k} (-1)^k k! \operatorname{Li}_{k+1}\left(\frac{z}{2}\right) \\
&= (-1)^q q! \sum_{k=0}^{q} \frac{\ln^{q-k}(2)}{(q-k)!} \operatorname{Li}_{k+1}\left(\frac{z}{2}\right).
\end{aligned}$$

Setting $z = 1$ yields

$$\int_0^{\frac{1}{2}} \frac{\ln^q(x)}{1-x} \mathrm{d}x = (-1)^q q! \sum_{k=0}^{q} \frac{\ln^{q-k}(2)}{(q-k)!} \operatorname{Li}_{k+1}\left(\frac{1}{2}\right). \tag{3.19}$$

Examples:

$$\int_0^{\frac{1}{2}} \frac{\ln(x)}{1-x} \mathrm{d}x = -\frac{1}{2}\zeta(2) - \frac{1}{2}\ln^2(2);$$

$$\int_0^{\frac{1}{2}} \frac{\ln^2(x)}{1-x} \mathrm{d}x = \frac{7}{4}\zeta(3) + \frac{1}{3}\ln^3(2);$$

$$\int_0^{\frac{1}{2}} \frac{\ln^3(x)}{1-x} \mathrm{d}x = \frac{3}{2}\ln^2(2)\zeta(2) - \frac{21}{4}\ln(2)\zeta(3) - \frac{1}{2}\ln^4(2) - 6\operatorname{Li}_4\left(\frac{1}{2}\right);$$

$$\int_0^{\frac{1}{2}} \frac{\ln^4(x)}{1-x} \mathrm{d}x = \frac{21}{2}\ln^2(2)\zeta(3) - 4\ln^3(2)\zeta(2) + \ln^5(2) + 24\ln(2)\operatorname{Li}_4\left(\frac{1}{2}\right)$$

$$+24\,\mathrm{Li}_5\left(\frac{1}{2}\right).$$

3.1.8 $\int_0^1 \frac{\ln^q(1+x)}{x}\mathrm{d}x$

Let $q \in \mathbb{Z}^+$. Then the following identity holds:

$$\int_0^1 \frac{\ln^q(1+x)}{x}\mathrm{d}x = \frac{\ln^{q+1}(2)}{q+1} + q!\,\zeta(q+1) - q!\sum_{k=0}^{q}\frac{\ln^{q-k}(2)}{(q-k)!}\,\mathrm{Li}_{k+1}\left(\frac{1}{2}\right). \tag{3.20}$$

The following proof may be found in [2, p. 4]:
Proof. Putting $\frac{1}{1+x} = y$,

$$\int_0^1 \frac{\ln^q(1+x)}{x}\mathrm{d}x = (-1)^q\int_{\frac{1}{2}}^1 \frac{\ln^q(y)}{y(1-y)}\mathrm{d}y$$

$$= (-1)^q\int_{\frac{1}{2}}^1 \frac{\ln^q(y)}{y}\mathrm{d}y + (-1)^q\underbrace{\int_{\frac{1}{2}}^1 \frac{\ln^q(y)}{1-y}\mathrm{d}y}_{\int_0^1 - \int_0^{1/2}}$$

$$= (-1)^q\left((-1)^q\frac{\ln^{q+1}(2)}{q+1}\right) + (-1)^q\left(\int_0^1 \frac{\ln^q(y)}{1-y}\mathrm{d}y - \int_0^{\frac{1}{2}}\frac{\ln^q(y)}{1-y}\mathrm{d}y\right).$$

The proof follows on substituting the results (3.8) and (3.19).
Examples:

$$\int_0^1 \frac{\ln(1+x)}{x}\mathrm{d}x = \frac{1}{2}\zeta(2);$$

$$\int_0^1 \frac{\ln^2(1+x)}{x}\mathrm{d}x = \frac{1}{4}\zeta(3);$$

$$\int_0^1 \frac{\ln^3(1+x)}{x}\mathrm{d}x = 6\zeta(4) - \frac{21}{4}\ln(2)\zeta(3) + \frac{3}{2}\ln^2(2)\zeta(2) - \frac{1}{4}\ln^4(2)$$
$$-6\,\mathrm{Li}_4\left(\frac{1}{2}\right);$$

$$\int_0^1 \frac{\ln^4(1+x)}{x}\mathrm{d}x = 24\zeta(5) - \frac{21}{2}\ln^2(2)\zeta(3) + 4\ln^3(2)\zeta(2) - \frac{4}{5}\ln^5(2)$$
$$-24\ln(2)\,\mathrm{Li}_4\left(\frac{1}{2}\right) - 24\,\mathrm{Li}_5\left(\frac{1}{2}\right).$$

3.1.9 $\int_0^\infty \frac{\ln^{2q}(x)}{1+x^2}\mathrm{d}x$

Let $q \in \mathbb{Z}^+$. Then the following identity holds:

$$\int_0^\infty \frac{\ln^{2q}(x)}{1+x^2}\mathrm{d}x = |E_{2q}|\left(\frac{\pi}{2}\right)^{2q+1}. \tag{3.21}$$

Proof. Make the change of variable $x = \sqrt{y}$,

$$\begin{aligned}
\int_0^\infty \frac{\ln^{2q}(x)}{1+x^2}\mathrm{d}x &= 2^{-2q-1}\int_0^\infty \frac{y^{-\frac{1}{2}}\ln^{2q}(y)}{1+y}\mathrm{d}y\\
&= 2^{-2q-1}\lim_{m\to\frac{1}{2}}\int_0^\infty \frac{y^{m-1}\ln^{2q}(y)}{1+y}\mathrm{d}y\\
&= 2^{-2q-1}\lim_{m\to\frac{1}{2}}\int_0^\infty \frac{\partial^{2q}}{\partial m^{2q}}\frac{y^{m-1}}{1+y}\mathrm{d}y
\end{aligned}$$

$\{$ use differentiation under the integral sign theorem (2.64)$\}$

$$= 2^{-2q-1}\lim_{m\to\frac{1}{2}}\frac{d^{2q}}{dm^{2q}}\int_0^\infty \frac{y^{m-1}}{1+y}\mathrm{d}y$$

$\{$set $a = m$ and $b = 1 - m$ in the beta function (1.50)$\}$

$$= 2^{-2q-1}\lim_{m\to\frac{1}{2}}\frac{d^{2q}}{dm^{2q}}\Gamma(m)\Gamma(1-m)$$

$\{$recall Euler's reflection formula (1.40)$\}$

$$= 2^{-2q-1}\pi\lim_{m\to\frac{1}{2}}\frac{d^{2q}}{dm^{2q}}\csc(m\pi)$$

$\{$this limit is given in (1.168)$\}$

$$\begin{aligned}
&= 2^{-2q-1}\pi|E_{2q}|\pi^{2q}\\
&= |E_{2q}|\left(\frac{\pi}{2}\right)^{2q+1}.
\end{aligned}$$

Examples:

$$\begin{aligned}
\int_0^\infty \frac{\ln^2(x)}{1+x^2}\mathrm{d}x &= \frac{\pi^3}{8};\\
\int_0^\infty \frac{\ln^4(x)}{1+x^2}\mathrm{d}x &= \frac{5\pi^5}{32};\\
\int_0^\infty \frac{\ln^6(x)}{1+x^2}\mathrm{d}x &= \frac{61\pi^7}{128};
\end{aligned}$$

$$\int_0^\infty \frac{\ln^8(x)}{1+x^2}\mathrm{d}x = \frac{1385\pi^9}{512}.$$

For the odd case $(2q+1)$, let $x \to \frac{1}{x}$

$$\begin{aligned}\int_0^\infty \frac{\ln^{2q+1}(x)}{1+x^2}\mathrm{d}x &= -\int_0^\infty \frac{\ln^{2q+1}(x)}{1+x^2}\mathrm{d}x \\ &\{\text{add the integral to both sides then divide by } 2\} \\ &= 0. \end{aligned} \tag{3.22}$$

3.1.10 $\int_0^\infty \frac{\ln^q(1+x)}{1+x^2}\mathrm{d}x$

Let $q \in \mathbb{Z}^+$. Then the following identity holds:

$$\int_0^\infty \frac{\ln^q(1+x)}{1+x^2}\mathrm{d}x = q!\,\Im\{\operatorname{Li}_{q+1}(1+i)\}. \tag{3.23}$$

The following proof may be found in [34]:

Proof. Let $x = \frac{1}{y}$,

$$\begin{aligned}\int_0^\infty \frac{\ln^q(1+x)}{1+x^2}\mathrm{d}x &= \int_0^\infty \frac{\ln^q\left(\frac{1+y}{y}\right)}{1+y^2}\mathrm{d}y \\ &\overset{\frac{y}{1+y}=x}{=} (-1)^q \int_0^1 \frac{\ln^q(x)}{x^2+(1-x)^2}\mathrm{d}x \\ &\left\{\text{write } \frac{1}{x^2+(1-x)^2} = \Im\frac{1+i}{1-(1+i)x}\right\} \\ &= (-1)^q \Im\int_0^1 \frac{(1+i)\ln^q(x)}{1-(1+i)x}\mathrm{d}x \\ &\{\text{set } z = 1+i \text{ in } (1.75)\} \\ &= q!\,\Im\{\operatorname{Li}_{q+1}(1+i)\}.\end{aligned}$$

Notice that the first line of the proof shows

$$\int_0^\infty \frac{\ln^q\left(\frac{1+x}{x}\right)}{1+x^2}\mathrm{d}x = q!\,\Im\{\operatorname{Li}_{q+1}(1+i)\}.$$

3.1.11 $\int_0^1 \frac{\ln^q\left(\frac{1+x}{x}\right)}{1+x^2}\mathrm{d}x$

Let $q \in \mathbb{Z}^+$. Then the following identity holds:

$$\int_0^1 \frac{\ln^q\left(\frac{1+x}{x}\right)}{1+x^2}\mathrm{d}x = q!\sum_{k=0}^{q}\frac{\ln^{q-k}(2)}{(q-k)!}\,\Im\left\{\mathrm{Li}_{k+1}\left(\frac{1+i}{2}\right)\right\}. \tag{3.24}$$

Proof. With $y = \frac{x}{1+x}$, we have

$$\begin{aligned}\int_0^1 \frac{\ln^q\left(\frac{1+x}{x}\right)}{1+x^2}\mathrm{d}x &= (-1)^q\int_0^{\frac{1}{2}} \frac{\ln^q(y)}{y^2+(1-y)^2}\mathrm{d}y \\ &= (-1)^q\,\Im\int_0^{\frac{1}{2}}\frac{(1+i)\ln^q(y)}{1-(1+i)y}\mathrm{d}y.\end{aligned}$$

The proof completes on setting $z = 1+i$ in (3.18).

3.1.12 $\int_0^1 \frac{\ln^q(1-x)}{1+x^2}\mathrm{d}x$

Let $q \in \mathbb{Z}^+$. Then the following identity holds:

$$\int_0^1 \frac{\ln^q(1-x)}{1+x^2}\mathrm{d}x = (-1)^q q!\,\Im\left\{\mathrm{Li}_{q+1}\left(\frac{1+i}{2}\right)\right\}. \tag{3.25}$$

Proof. Let $1-x = y$,

$$\begin{aligned}\int_0^1 \frac{\ln^q(1-x)}{1+x^2}\mathrm{d}x &= \int_0^1 \frac{\ln^q(y)}{1+(1-y)^2}\mathrm{d}y \\ &\left\{\text{write } \frac{1}{1+(1-y)^2} = \Im\frac{i}{1+i-iy}\right\} \\ &= \Im\int_0^1\frac{i\ln^q(y)}{1+i-iy}\mathrm{d}y \\ &\{\text{ to get this integral, set } z=-i \text{ in (1.76)}\} \\ &= (-1)^q q!\,\Im\left\{\mathrm{Li}_{q+1}\left(\frac{-i}{-i-1}\right)\right\} \\ &= (-1)^q q!\,\Im\left\{\mathrm{Li}_{q+1}\left(\frac{1+i}{2}\right)\right\}.\end{aligned}$$

With $q = 2$, we have

$$\int_0^1 \frac{\ln^2(1-x)}{1+x^2}\mathrm{d}x = 2\,\Im\left\{\operatorname{Li}_3\left(\frac{1+i}{2}\right)\right\}$$

$$\{\text{recall the relation (1.103)}\}$$

$$= \frac{7\pi^3}{64} + \frac{3\pi}{16}\ln^2(2) - 2\,\Im\{\operatorname{Li}_3(1+i)\}. \tag{3.26}$$

Applying the same approach, we also have

$$\int_0^1 \frac{x\ln^q(1-x)}{1+x^2}\mathrm{d}x \overset{1-x=y}{=} \int_0^1 \frac{(1-y)\ln^q(y)}{1+(1-y)^2}\mathrm{d}y$$

$$\left\{\text{write } \frac{1-y}{1+(1-y)^2} = \Re\frac{i}{1+i-iy}\right\}$$

$$= \Re\int_0^1 \frac{i\ln^q(y)}{1+i-iy}\mathrm{d}y$$

$$= (-1)^q q!\,\Re\left\{\operatorname{Li}_{q+1}\left(\frac{1+i}{2}\right)\right\}. \tag{3.27}$$

3.1.13 $\int_0^1 \frac{\ln^q(1+x)}{1+x^2}\mathrm{d}x$

Let $q \in \mathbb{Z}^+$. Then the following identity holds:

$$\int_0^1 \frac{\ln^q(1+x)}{1+x^2}\mathrm{d}x = q!\,\Im\left(\operatorname{Li}_{q+1}(1+i) - \sum_{k=0}^{q}\frac{\ln^{q-k}(2)}{(q-k)!}\operatorname{Li}_{k+1}\left(\frac{1+i}{2}\right)\right). \tag{3.28}$$

Proof. With $y = \frac{1}{1+x}$, we have

$$\int_0^1 \frac{\ln^q(1+x)}{1+x^2}\mathrm{d}x = (-1)^q\int_{\frac{1}{2}}^1 \frac{\ln^q(y)}{y^2+(1-y)^2}\mathrm{d}y$$

$$= (-1)^q\Im\underbrace{\int_{\frac{1}{2}}^1 \frac{(1+i)\ln^q(y)}{1-(1+i)y}\mathrm{d}y}_{\int_0^1 - \int_0^{1/2}}$$

$$= (-1)^q\,\Im\left(\int_0^1 \frac{(1+i)\ln^q(y)}{1-(1+i)y}\mathrm{d}y - \int_0^{\frac{1}{2}} \frac{(1+i)\ln^q(y)}{1-(1+i)y}\mathrm{d}y\right).$$

The proof completes on setting $z=1+i$ in (1.75) and (3.18).
With $q=2$, we have

$$\begin{aligned}\int_0^1 \frac{\ln^2(1+x)}{1+x^2}\mathrm{d}x &= -2\ln(2)G+2\,\Im\left\{\operatorname{Li}_3(1+i)-\operatorname{Li}_3\left(\frac{1+i}{2}\right)\right\}\\ &\qquad\{\text{recall the relation (1.103)}\}\\ &= 4\,\Im\{\operatorname{Li}_3(1+i)\}-\frac{7\pi^3}{64}-\frac{3\pi}{16}\ln^2(2)-2\ln(2)G. \end{aligned}\tag{3.29}$$

Applying the same approach, we get

$$\begin{aligned}\int_0^1 \frac{x\ln^q(1+x)}{1+x^2}\mathrm{d}x &= (-1)^q\int_{\frac{1}{2}}^1 \frac{\ln^q(y)}{y}\mathrm{d}y+(-1)^q\int_{\frac{1}{2}}^1 \frac{(1-2y)\ln^q(y)}{y^2+(1-y)^2}\mathrm{d}y\\ &\left\{\text{write } \frac{1-2y}{y^2+(1-y)^2}=\Re\frac{1+i}{1-(1+i)y} \text{ in the second integral}\right\}\\ &= \frac{\ln^{q+1}(2)}{q+1}+(-1)^q\Re\int_{\frac{1}{2}}^1 \frac{(1+i)\ln^q(y)}{1-(1+i)y}\mathrm{d}y\\ &= \frac{\ln^{q+1}(2)}{q+1}+(-1)^q\Re\left(\int_0^1 \frac{(1+i)\ln^q(y)}{1-(1+i)y}\mathrm{d}y-\int_0^{\frac{1}{2}} \frac{(1+i)\ln^q(y)}{1-(1+i)y}\mathrm{d}y\right)\\ &\qquad\{\text{set } z=1+i \text{ in (1.75) and (3.18) to get these two integrals }\}\\ &= \frac{\ln^{q+1}(2)}{q+1}+q!\,\Re\left(\operatorname{Li}_{q+1}(1+i)-\sum_{k=0}^{q}\frac{\ln^{q-k}(2)}{(q-k)!}\operatorname{Li}_{k+1}\left(\frac{1+i}{2}\right)\right). \end{aligned}\tag{3.30}$$

3.1.14 $\int_0^1 \frac{\ln^{q-1}\left(\frac{x}{1-x}\right)}{1+x}\mathrm{d}x$

Let $q\in\mathbb{Z}_{>1}$. Then the following identity holds:

$$\begin{aligned}\int_0^1 \frac{\ln^{q-1}(\frac{x}{1-x})}{1+x}\mathrm{d}x &= (1+(-1)^q)(q-1)!\,\eta(q)\\ &\quad -2(-1)^q(q-1)!\sum_{k=0}^{\lfloor\frac{q}{2}\rfloor}\frac{\eta(2k)}{(q-2k)!}\ln^{q-2k}(2). \end{aligned}\tag{3.31}$$

Proof. Start with the change of variable $\frac{x}{1-x}=y$,

$$\int_0^1 \frac{\ln^{q-1}(\frac{x}{1-x})}{1+x}\mathrm{d}x=\int_0^\infty \frac{\ln^{q-1}(y)}{(1+y)(1+2y)}\mathrm{d}y$$

$$= \int_0^1 \frac{\ln^{q-1}(y)}{(1+y)(1+2y)}\mathrm{d}y + \underbrace{\int_1^\infty \frac{\ln^{q-1}(y)}{(1+y)(1+2y)}\mathrm{d}y}_{y\to 1/y}$$

$$= \int_0^1 \frac{\ln^{q-1}(y)}{(1+y)(1+2y)}\mathrm{d}y - (-1)^q \int_0^1 \frac{\ln^{q-1}(y)}{(1+y)(2+y)}\mathrm{d}y$$

{decompose both integrands}

$$= \int_0^1 \frac{2\ln^{q-1}(y)}{1+2y}\mathrm{d}y + (-1)^q \int_0^1 \frac{\frac{1}{2}\ln^{q-1}(y)}{1+\frac{1}{2}y}\mathrm{d}y - (1+(-1)^q)\int_0^1 \frac{\ln^{q-1}(y)}{1+y}\mathrm{d}y$$

{use (1.75) for the first two integrals and (1.53) for the last}

$$= (-1)^q(q-1)!\left(\operatorname{Li}_q(-2) + (-1)^q \operatorname{Li}_q\left(-\frac{1}{2}\right)\right) + (1+(-1)^q)(q-1)!\,\eta(q).$$

The proof finishes on setting $z = 2$ in (1.99).

Examples:

$$\int_0^1 \frac{\ln(\frac{x}{1-x})}{1+x}\mathrm{d}x = -\frac{1}{2}\ln^2(2);$$

$$\int_0^1 \frac{\ln^2(\frac{x}{1-x})}{1+x}\mathrm{d}x = 2\ln(2)\zeta(2) + \frac{1}{3}\ln^3(2);$$

$$\int_0^1 \frac{\ln^3(\frac{x}{1-x})}{1+x}\mathrm{d}x = -3\ln^2(2)\zeta(2) - \frac{1}{4}\ln^4(2);$$

$$\int_0^1 \frac{\ln^4(\frac{x}{1-x})}{1+x}\mathrm{d}x = 42\ln(2)\zeta(4) + 4\ln^3(2)\zeta(2) + \frac{1}{5}\ln^5(2).$$

3.1.15 $\int_0^\infty \frac{x^{s-1}\ln(1+x)}{1+x}\mathrm{d}x$

Let $-1 < \Re(s) < 1$. Then the following identity holds:

$$\int_0^\infty \frac{x^{s-1}\ln(1+x)}{1+x}\mathrm{d}x = -\pi\csc(\pi s)\left(\psi(1-s)+\gamma\right). \tag{3.32}$$

The following proof may be found in [16]:

Proof.

$$\int_0^\infty \frac{x^{s-1}\ln(1+x)}{1+x}\mathrm{d}x = \lim_{r\to 1}\int_0^\infty \frac{x^{s-1}\ln(1+x)}{(1+x)^r}\mathrm{d}x$$

$$= \lim_{r\to 1}\int_0^\infty -\frac{\partial}{\partial r}\frac{x^{s-1}}{(1+x)^r}\mathrm{d}x$$

$$\{\text{ use differentiation under the integral sign theorem (2.63)}\}$$
$$= -\lim_{r\to 1}\frac{\partial}{\partial r}\int_0^\infty \frac{x^{s-1}}{(1+x)^r}\mathrm{d}x$$
$$\{\text{let } a = s \text{ and } b = r-s \text{ in the beta function (1.51)}\}$$
$$= -\lim_{r\to 1}\frac{\partial}{\partial r}\,\mathrm{B}(s, r-s)$$
$$= -\lim_{r\to 1}\frac{\partial}{\partial r}\frac{\Gamma(s)\Gamma(r-s)}{\Gamma(r)}$$
$$\{\text{use } \Gamma'(x) = \Gamma(x)\psi(x) \text{ given in (1.144)}\}$$
$$= -\Gamma(s)\lim_{r\to 1}\frac{\Gamma(r-s)(\psi(r-s)-\psi(r))}{\Gamma(r)}$$
$$= -\Gamma(s)\Gamma(1-s)(\psi(1-s)-\psi(1))$$
$$\{\text{recall Euler reflection formula (1.40) and write } \psi(1) = -\gamma\}$$
$$= -\pi\csc(\pi s)\left(\psi(1-s)+\gamma\right).$$

3.1.16 $\int_0^\infty \frac{\ln^{2q-1}(x)\ln(1+x)}{x(1+x)}\mathrm{d}x$

Let $q \in \mathbb{Z}^+$. Then the following identity holds:

$$\int_0^\infty \frac{\ln^{2q-1}(x)\ln(1+x)}{x(1+x)}\mathrm{d}x = 2(2q-1)!\sum_{k=0}^{q-1}\eta(2k)\zeta(2q-2k+1). \quad (3.33)$$

The following proof may be found in [18]:

Proof.

$$\int_0^\infty \frac{\ln^{2q-1}(x)\ln(1+x)}{x(1+x)}\mathrm{d}x = \lim_{s\to 0}\int_0^\infty \frac{x^{s-1}\ln^{2q-1}(x)\ln(1+x)}{1+x}\mathrm{d}x$$
$$= \lim_{s\to 0}\int_0^\infty \frac{\partial^{2q-1}}{\partial s^{2q-1}}\frac{x^{s-1}\ln(1+x)}{1+x}\mathrm{d}x$$
$$\{\text{ use differentiation under the integral sign theorem (2.64)}\}$$
$$= \lim_{s\to 0}\frac{d^{2q-1}}{ds^{2q-1}}\int_0^\infty \frac{x^{s-1}\ln(1+x)}{1+x}\mathrm{d}x$$
$$\{\text{this integral is given in (3.32)}\}$$
$$= -\lim_{s\to 0}\frac{d^{2q-1}}{ds^{2q-1}}\pi\csc(\pi s)\left(\psi(1-s)+\gamma\right)$$
$$\{\text{multiply by } s/s\}$$

$$= -\lim_{s\to 0}\frac{d^{2q-1}}{ds^{2q-1}}\left(\pi s\csc(\pi s)\right)\left(\frac{\psi(1-s)+\gamma}{s}\right)$$

{expand both functions in series as in (2.99) and (1.156)}

$$= -\lim_{s\to 0}\frac{d^{2q-1}}{ds^{2q-1}}\left(2\sum_{n=0}^{\infty}\eta(2n)\,s^{2n}\right)\left(-\sum_{n=0}^{\infty}\zeta(n+2)\,s^{n}\right)$$

{let $a_{2n}=\eta(2n)$ and $b_n=\zeta(n+2)$ in (2.67)}

$$= 2\lim_{s\to 0}\frac{d^{2q-1}}{ds^{2q-1}}\sum_{n=0}^{\infty}\left(\underbrace{\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\eta(2k)\zeta(n-2k+2)}_{f_n}\right)s^n$$

$$= 2\lim_{s\to 0}\frac{d^{2q-1}}{ds^{2q-1}}\sum_{n=0}^{\infty}f_n s^n$$

$$\left\{\text{use }\frac{d^a}{ds^a}s^n=\frac{n!}{(n-a)!}s^{n-a},\quad n\ge a\right\}$$

$$= 2\lim_{s\to 0}\sum_{n=2q-1}^{\infty}\frac{n!}{(n-2q+1)!}f_n\,s^{n-2q+1}$$

$$= 2\lim_{s\to 0}\left((2q-1)!f_{2q-1}+(2q)!f_{2q}\,s+\frac{(2q+1)!}{2}f(2q+1)s^2+\cdots\right)$$

$$= 2(2q-1)!f_{2q-1}$$

$$= 2(2q-1)!\sum_{k=0}^{\lfloor\frac{2q-1}{2}\rfloor}\eta(2k)\zeta(2q-2k+1)$$

$$= 2(2q-1)!\sum_{k=0}^{q-1}\eta(2k)\zeta(2q-2k+1).$$

Examples:

$$\int_0^\infty\frac{\ln(x)\ln(1+x)}{x(1+x)}\mathrm{d}x=\zeta(3);$$

$$\int_0^\infty\frac{\ln^3(x)\ln(1+x)}{x(1+x)}\mathrm{d}x=6\zeta(5)+6\zeta(2)\zeta(3);$$

$$\int_0^\infty\frac{\ln^5(x)\ln(1+x)}{x(1+x)}\mathrm{d}x=120\zeta(7)+120\zeta(2)\zeta(5)+210\zeta(3)\zeta(4);$$

$$\int_0^\infty\frac{\ln^7(x)\ln(1+x)}{x(1+x)}\mathrm{d}x=5040\zeta(9)+5040\zeta(2)\zeta(7)+9765\zeta(3)\zeta(6)$$
$$+8820\zeta(4)\zeta(5).$$

3.1.17 $\int_0^\infty \frac{\ln^{2q}(x)\ln(1+x^2)}{1+x^2}\mathrm{d}x$

The following identity may be found in [17]:

$$\int_0^\infty \frac{\ln^{2q}(x)\ln(1+x^2)}{1+x^2}\mathrm{d}x = 2\ln(2)|E_{2q}|\left(\frac{\pi}{2}\right)^{2q+1}$$

$$+\left(\frac{\pi}{2}\right)^{2q+1}\sum_{k=1}^{q}\binom{2q}{2k}\frac{(2k)!}{\pi^{2k}}|E_{2q-2k}|(2^{2k+1}-1)\zeta(2k+1). \tag{3.34}$$

Proof.

$$\int_0^\infty \frac{\ln^{2q}(x)\ln(1+x)}{\sqrt{x}(1+x)}\mathrm{d}x = \lim_{s\to\frac{1}{2}}\int_0^\infty \frac{x^{s-1}\ln^{2q}(x)\ln(1+x)}{1+x}\mathrm{d}x$$

$$= \lim_{s\to\frac{1}{2}}\int_0^\infty \frac{\partial^{2q}}{\partial s^{2q}}\frac{x^{s-1}\ln(1+x)}{1+x}\mathrm{d}x$$

$$= \lim_{s\to\frac{1}{2}}\frac{d^{2q}}{ds^{2q}}\int_0^\infty \frac{x^{s-1}\ln(1+x)}{1+x}\mathrm{d}x$$

$$\{\text{this integral is given in (3.32)}\}$$

$$= -\pi\lim_{s\to\frac{1}{2}}\frac{d^{2q}}{ds^{2q}}\csc(s\pi)\left(\psi(1-s)+\gamma\right)$$

$$\left\{\text{use } \frac{d^q}{ds^q}(f(s)*g(s)) = \sum_{k=0}^{q}\binom{q}{k}\frac{d^{q-k}}{ds^{q-k}}f(s)*\frac{d^k}{ds^k}g(s)\right\}$$

$$= -\pi\sum_{k=0}^{2q}\binom{2q}{k}\lim_{s\to\frac{1}{2}}\frac{d^{2q-k}}{ds^{2q-k}}\csc(s\pi)\lim_{s\to\frac{1}{2}}\frac{d^k}{ds^k}(\psi(1-s)+\gamma)$$

$$\left\{\text{use } \sum_{k=0}^{2q}f(k) = \sum_{k=0}^{q}f(2k+1)+\sum_{k=0}^{q}f(2k)\right\}$$

$$= -\pi\sum_{k=0}^{q}\binom{2q}{2k+1}\lim_{s\to\frac{1}{2}}\frac{d^{2q-2k-1}}{ds^{2q-2k-1}}\csc(s\pi)\lim_{s\to\frac{1}{2}}\frac{d^{2k+1}}{ds^{2k+1}}(\psi(1-s)+\gamma)$$

$$-\pi\sum_{k=0}^{q}\binom{2q}{2k}\lim_{s\to\frac{1}{2}}\frac{d^{2q-2k}}{ds^{2q-2k}}\csc(s\pi)\lim_{s\to\frac{1}{2}}\frac{d^{2k}}{ds^{2k}}(\psi(1-s)+\gamma)$$

$$\left\{\text{ignore the first sum, since } \lim_{s\to\frac{1}{2}}\frac{d^p}{ds^p}\csc(s\pi) = 0 \text{ for odd } p\right\}$$

$$\{\text{and use the result (1.168) for the first limit in the second sum}\}$$

$$= -\pi \sum_{k=0}^{q} \binom{2q}{2k} |E_{2q-2k}| \pi^{2q-2k} \lim_{s\to\frac{1}{2}} \frac{d^{2k}}{ds^{2k}} (\psi(1-s)+\gamma)$$

$$\{\text{separate the first term using } \psi(1/2)+\gamma = -2\ln(2) \text{ given in (1.148)}\}$$

$$= 2\ln(2)|E_{2q}|\pi^{2q+1}$$

$$-\pi \sum_{k=1}^{q} \binom{2q}{2k} |E_{2q-2k}| \pi^{2q-2k} \lim_{s\to\frac{1}{2}} \frac{d^{2k}}{ds^{2k}} (\psi(1-s)+\gamma)$$

$$\{\text{use the definition of the polygamma function (1.150)}\}$$

$$= 2\ln(2)|E_{2q}|\pi^{2q+1}$$

$$-\pi \sum_{k=1}^{q} \binom{2q}{2k} |E_{2q-2k}| \pi^{2q-2k} \psi^{(2k)}\left(\frac{1}{2}\right)$$

$$\left\{\text{substitute the result of } \psi^{(q)}(1/2) \text{ given in (1.157)}\right\}$$

$$= 2\ln(2)|E_{2q}|\pi^{2q+1}$$

$$+\pi \sum_{k=1}^{q} \binom{2q}{2k} |E_{2q-2k}| \pi^{2q-2k} \, (2k)!(2^{2k+1}-1)\zeta(2k+1).$$

The proof completes on setting $\sqrt{x}=y$ in the integral:

$$\int_0^\infty \frac{\ln^{2q}(x)\ln(1+x)}{\sqrt{x}(1+x)} dx = 2^{2q+1} \int_0^\infty \frac{\ln^{2q}(y)\ln(1+y^2)}{1+y^2} dy.$$

Examples:

$$\int_0^\infty \frac{\ln^2(x)\ln(1+x^2)}{1+x^2} dx = \frac{\pi^3}{4}\ln(2) + \frac{7\pi}{4}\zeta(3);$$

$$\int_0^\infty \frac{\ln^4(x)\ln(1+x^2)}{1+x^2} dx = \frac{5}{16}\pi^5\ln(2) + \frac{21}{8}\pi^3\zeta(3) + \frac{93}{4}\pi\zeta(5);$$

$$\int_0^\infty \frac{\ln^6(x)\ln(1+x^2)}{1+x^2} dx = \frac{61}{64}\pi^7\ln(2) + \frac{525}{64}\pi^5\zeta(3) + \frac{1395}{16}\pi^3\zeta(5) + \frac{5715}{8}\pi\zeta(7);$$

$$\int_0^\infty \frac{\ln^8(x)\ln(1+x^2)}{1+x^2} dx = \frac{1385}{256}\pi^9\ln(2) + \frac{2989}{64}\pi^7\zeta(3) + \frac{16275}{32}\pi^5\zeta(5) + \frac{40005}{8}\pi^3\zeta(7) + \frac{160965}{4}\pi\zeta(9).$$

3.1.18 $\int_0^1 \frac{\ln^{2q}(x)\ln(1+x^2)}{1+x^2}\mathrm{d}x$

Let $q \in \mathbb{Z}_{\geq 0}$. Then the following identity holds:

$$\begin{aligned}\int_0^1 \frac{\ln^{2q}(x)\ln(1+x^2)}{1+x^2}\mathrm{d}x &= -(2q+1)!\,\beta(2q+2) + \ln(2)|E_{2q}|\left(\frac{\pi}{2}\right)^{2q+1} \\ &+\frac{1}{2}\left(\frac{\pi}{2}\right)^{2q+1}\sum_{k=1}^{q}\binom{2q}{2k}\frac{(2k)!}{\pi^{2k}}|E_{2q-2k}|(2^{2k+1}-1)\zeta(2k+1). \qquad (3.35)\end{aligned}$$

Proof.

$$\begin{aligned}&\int_0^1 \frac{\ln^{2q}(x)\ln(1+x^2)}{1+x^2}\mathrm{d}x \\ &= \int_0^\infty \frac{\ln^{2q}(x)\ln(1+x^2)}{1+x^2}\mathrm{d}x - \underbrace{\int_1^\infty \frac{\ln^{2q}(x)\ln(1+x^2)}{1+x^2}\mathrm{d}x}_{x\to 1/x} \\ &= \int_0^\infty \frac{\ln^{2q}(x)\ln(1+x^2)}{1+x^2}\mathrm{d}x - \int_0^1 \frac{\ln^{2q}(x)\ln(1+x^2)}{1+x^2}\mathrm{d}x \\ &\quad +2\int_0^1 \frac{\ln^{2q+1}(x)}{1+x^2}\mathrm{d}x \\ &\left\{\text{add} \int_0^1 \frac{\ln^{2q}(x)\ln(1+x^2)}{1+x^2}\mathrm{d}x \text{ to both sides then divide by } 2\right\} \\ &= \frac{1}{2}\int_0^\infty \frac{\ln^{2q}(x)\ln(1+x^2)}{1+x^2}\mathrm{d}x + \int_0^1 \frac{\ln^{2q+1}(x)}{1+x^2}\mathrm{d}x.\end{aligned}$$

These two integrals are given in (3.34) and (1.62) respectively.

Examples:

$$\int_0^1 \frac{\ln^2(x)\ln(1+x^2)}{1+x^2}\mathrm{d}x = \frac{\pi^3}{8}\ln(2) + \frac{7\pi}{8}\zeta(3) - 6\beta(4);$$

$$\int_0^1 \frac{\ln^4(x)\ln(1+x^2)}{1+x^2}\mathrm{d}x = \frac{5}{32}\pi^5\ln(2) + \frac{21}{16}\pi^3\zeta(3) + \frac{93}{8}\pi\zeta(5) - 120\beta(6);$$

$$\begin{aligned}\int_0^1 \frac{\ln^6(x)\ln(1+x^2)}{1+x^2}\mathrm{d}x &= \frac{61}{128}\pi^7\ln(2) + \frac{525}{128}\pi^5\zeta(3) + \frac{1395}{32}\pi^3\zeta(5) \\ &+\frac{5715}{16}\pi\zeta(7) - 5040\beta(8);\end{aligned}$$

$$\int_0^1 \frac{\ln^8(x)\ln(1+x^2)}{1+x^2}\mathrm{d}x = \frac{1385}{512}\pi^9\ln(2) + \frac{2989}{128}\pi^7\zeta(3) + \frac{16275}{64}\pi^5\zeta(5)$$

$$+\frac{40005}{16}\pi^3\zeta(7)+\frac{160965}{8}\pi\zeta(9)-362880\beta(10).$$

3.1.19 $\int_0^1 \frac{\ln^{2q-1}(x)\ln(1+x)}{x(1+x)}\mathrm{d}x$

Let $q \in \mathbb{Z}^+$. Then the following identity holds:

$$\int_0^1 \frac{\ln^{2q-1}(x)\ln(1+x)}{x(1+x)}\mathrm{d}x = -\frac{1}{2}(2q+1)(2q-1)!\,\eta(2q+1)$$
$$+(2q-1)!\sum_{k=0}^{q-1}\eta(2k)\zeta(2q-2k+1). \tag{3.36}$$

Proof.

$$\int_0^1 \frac{\ln^{2q-1}(x)\ln(1+x)}{x(1+x)}\mathrm{d}x$$
$$=\int_0^\infty \frac{\ln^{2q-1}(x)\ln(1+x)}{x(1+x)}\mathrm{d}x-\underbrace{\int_1^\infty \frac{\ln^{2q-1}(x)\ln(1+x)}{x(1+x)}\mathrm{d}x}_{x\mapsto 1/x}$$
$$=\int_0^\infty \frac{\ln^{2q-1}(x)\ln(1+x)}{x(1+x)}\mathrm{d}x+\int_0^1 \frac{\ln^{2q-1}(x)\ln\left(\frac{1+x}{x}\right)}{1+x}\mathrm{d}x$$
$$=\int_0^\infty \frac{\ln^{2q-1}(x)\ln(1+x)}{x(1+x)}\mathrm{d}x+\int_0^1 \frac{\ln^{2q-1}(x)\ln(1+x)}{1+x}\mathrm{d}x-\int_0^1 \frac{\ln^{2q}(x)}{1+x}\mathrm{d}x.$$

By adding

$$\int_0^1 \frac{\ln^{2q-1}(x)\ln(1+x)}{x(1+x)}\mathrm{d}x$$
$$=\underbrace{\int_0^1 \frac{\ln^{2q-1}(x)\ln(1+x)}{x}\mathrm{d}x}_{\text{IBP}}-\int_0^1 \frac{\ln^{2q-1}(x)\ln(1+x)}{1+x}\mathrm{d}x$$
$$=-\frac{1}{2q}\int_0^1 \frac{\ln^{2q}(x)}{1+x}\mathrm{d}x-\int_0^1 \frac{\ln^{2q-1}(x)\ln(1+x)}{1+x}\mathrm{d}x$$

to both sides, the integral $\int_0^1 \frac{\ln^{2q-1}(x)\ln(1+x)}{1+x}\mathrm{d}x$ nicely cancels out,

$$2\int_0^1 \frac{\ln^{2q-1}(x)\ln(1+x)}{x(1+x)}\mathrm{d}x$$

$$= \int_0^\infty \frac{\ln^{2q-1}(x)\ln(1+x)}{x(1+x)}\mathrm{d}x - \frac{2q+1}{2q}\int_0^1 \frac{\ln^{2q}(x)}{1+x}\mathrm{d}x. \tag{3.37}$$

These two integrals are given in (3.33) and (1.53) respectively.

Examples:

$$\int_0^1 \frac{\ln(x)\ln(1+x)}{x(1+x)}\mathrm{d}x = -\frac{5}{8}\zeta(3);$$

$$\int_0^1 \frac{\ln^3(x)\ln(1+x)}{x(1+x)}\mathrm{d}x = -\frac{177}{16}\zeta(5) + 3\zeta(2)\zeta(3);$$

$$\int_0^1 \frac{\ln^5(x)\ln(1+x)}{x(1+x)}\mathrm{d}x = -\frac{5655}{16}\zeta(7) + 60\zeta(2)\zeta(5) + 105\zeta(3)\zeta(4);$$

$$\begin{aligned}\int_0^1 \frac{\ln^7(x)\ln(1+x)}{x(1+x)}\mathrm{d}x = -\frac{642285}{32}\zeta(9) &+ 2520\zeta(2)\zeta(7) + 4410\zeta(4)\zeta(5)\\ &+\frac{9765}{2}\zeta(3)\zeta(6).\end{aligned}$$

The following integral is related:

$$\begin{aligned}&\int_0^1 \frac{\ln^{2q-1}(x)\ln(1+x)}{1+x}\mathrm{d}x\\
&= \int_0^1 \frac{\ln^{2q-1}(x)\ln(1+x)}{x}\mathrm{d}x - \int_0^1 \frac{\ln^{2q-1}(x)\ln(1+x)}{x(1+x)}\mathrm{d}x\\
&\qquad \{\text{these two integrals are given in (3.36) and (3.12)}\}\\
&= \frac{1}{2}(2q-1)!\left((2q-1)\eta(2q+1) - \sum_{k=0}^{q-1}\eta(2k)\zeta(2q-2k+1)\right).\end{aligned} \tag{3.38}$$

3.1.20 $\int_0^1 \frac{x^{s-1}\ln(x)\ln(1-x)}{1-x}\mathrm{d}x$

Let $\Re(s) > -1$. Then the following identity holds:

$$\int_0^1 \frac{x^{s-1}\ln(x)\ln(1-x)}{1-x}\mathrm{d}x = \psi^{(1)}(s)(\psi(s)+\gamma) - \frac{1}{2}\psi^{(2)}(s). \tag{3.39}$$

Proof.

$$\int_0^1 \frac{x^{s-1}\ln(x)\ln(1-x)}{1-x}\mathrm{d}x = \lim_{r\to 0}\int_0^1 \frac{x^{s-1}\ln(x)\ln(1-x)}{(1-x)^{1-r}}\mathrm{d}x$$

$$= \lim_{r\to 0} \int_0^1 \frac{\partial^2}{\partial s \partial r} x^{s-1}(1-x)^{r-1} \mathrm{d}x$$

$$= \lim_{r\to 0} \frac{\partial^2}{\partial s \partial r} \int_0^1 x^{s-1}(1-x)^{r-1} \mathrm{d}x$$

{recall the beta function (1.46)}

$$= \lim_{r\to 0} \frac{\partial^2}{\partial s \partial r} \mathrm{B}(s,r)$$

$$= \lim_{r\to 0} \frac{\partial^2}{\partial s \partial r} \frac{\Gamma(s)\Gamma(r)}{\Gamma(s+r)}$$

$$= \lim_{r\to 0} \frac{\Gamma(s)\Gamma(r)}{\Gamma(s+r)} \Big((\psi(s)-\psi(s+r))(\psi(r)-\psi(s+r)) - \psi^{(1)}(s+r) \Big)$$

$$\left\{ \text{write } \Gamma(r) = \frac{\Gamma(1+r)}{r} \text{ and } \psi(r) = \psi(1+r) - \frac{1}{r} \right\}$$

{given in (1.34) and (1.149) respectively}

$$= \lim_{r\to 0} \frac{\Gamma(s)\Gamma(1+r)}{\Gamma(s+r)} \times$$

$$\lim_{r\to 0} \frac{(\psi(s)-\psi(s+r))(\psi(1+r)-\frac{1}{r}-\psi(s+r)) - \psi^{(1)}(s+r)}{r}$$

$$\left\{ \text{multiply by } r/r \text{ and note that } \lim_{r\to 0} \frac{\Gamma(s)\Gamma(1+r)}{\Gamma(s+r)} = \frac{\Gamma(s)\Gamma(1)}{\Gamma(s)} = 1 \right\}$$

$$= \lim_{r\to 0} \frac{(\psi(s)-\psi(s+r))(r\psi(1+r)-1-r\psi(s+r)) - r\psi^{(1)}(s+r)}{r^2}$$

{now we can apply L'Hôpital's rule, since we have $0/0$}

$$= \lim_{r\to 0} \frac{1}{2r} \Big\{ \psi^{(1)}(s+r)\left[r\psi(s+r) - r\psi(1+r)\right] + \left[\psi(s)-\psi(s+r)\right]$$

$$\Big[\psi(1+r) - \psi(s+r) - \psi^{(1)}(s+r) + r\psi^{(1)}(1+r)\Big] - r\psi^{(2)}(s+r) \Big\}$$

{apply L'Hôpital's rule again, since we have $0/0$}

$$= \lim_{r\to 0} \frac{1}{2} \Big\{ 2\psi^{(1)}(s+r) \Big[\psi(s+r) + r\psi^{(1)}(s+r) - \psi(1+r) - r\psi^{(1)}(1+r)\Big]$$

$$+\psi^{(2)}(s+r)\left[r\psi(s+r) - b\psi(1+r) - 1\right] + \left[\psi(s)-\psi(s+r)\right]$$

$$\Big[2\psi^{(2)}(s+r) + r\psi^{(2)}(s+r) - 2\psi^{(2)}(1+r) - r\psi^{(2)}(1+r)\Big] - r\psi^{(3)}(s+r) \Big\}$$

$$= \frac{1}{2} \Big\{ 2\psi^{(1)}(s)(\psi(s)-\psi(1)) - \psi^{(2)}(s) \Big\}$$

{write $\psi(1) = -\gamma$ given in (1.147)}

$$= \psi^{(1)}(s)(\psi(s)+\gamma) - \frac{1}{2}\psi^{(2)}(s).$$

Remark: We know that the beta function, $\mathrm{B}(s,r)=\int_0^1 x^{s-1}(1-x)^{r-1}\mathrm{d}x$, is defined for $\Re(s)>0$. However, its derivative in (3.39) is defined for $\Re(s)>-1$ due to the analytic continuation (see[40]).

3.1.21 $\int_0^1 \frac{x^{s-1}\ln^{q-1}(x)\ln(1-x)}{1-x}\mathrm{d}x$

Let $s\notin\mathbb{Z}^-$ and $q\in\mathbb{Z}_{>1}$. Then the following identity holds:

$$\int_0^1 \frac{x^{s-1}\ln^{q-1}(x)\ln(1-x)}{1-x}\mathrm{d}x=\psi^{(q-1)}(s)(\psi(s)+\gamma)-\frac{1}{2}\psi^{(q)}(s)$$
$$+\sum_{k=1}^{q-2}\binom{q-2}{k}\psi^{(q-k-1)}(s)\psi^{(k)}(s). \tag{3.40}$$

Proof.

$$\begin{aligned}
\int_0^1 \frac{x^{s-1}\ln^{q-1}(x)\ln(1-x)}{1-x}\mathrm{d}x &= \int_0^1 \frac{\partial^{q-2}}{\partial s^{q-2}}\frac{x^{s-1}\ln(x)\ln(1-x)}{1-x}\mathrm{d}x\\
&= \frac{d^{q-2}}{ds^{q-2}}\int_0^1 \frac{x^{s-1}\ln(x)\ln(1-x)}{1-x}\mathrm{d}x\\
&\quad\{\text{this integral is given in (3.39)}\}\\
&= \frac{d^{q-2}}{ds^{q-2}}\left(-\frac{1}{2}\psi^{(2)}(s)+\psi^{(1)}(s)(\psi(s)+\gamma)\right)\\
&= -\frac{1}{2}\frac{d^{q-2}}{ds^{q-2}}\psi^{(q)}(s)+\frac{d^{q-2}}{ds^{q-2}}\psi^{(1)}(s)(\psi(s)+\gamma)\\
&\quad\left\{\text{use }\frac{d^q}{ds^q}(f(s)*g(s))=\sum_{k=0}^{q}\binom{q}{k}\frac{d^{q-k}}{ds^{q-k}}f(s)*\frac{d^k}{ds^k}g(s)\right\}\\
&= -\frac{1}{2}\frac{d^{q-2}}{ds^{q-2}}\psi^{(2)}(s)+\sum_{k=0}^{q-2}\binom{q-2}{k}\frac{d^{q-k-2}}{ds^{q-k-2}}\psi^{(1)}(s)\frac{d^k}{ds^k}(\psi(s)+\gamma)\\
&= -\frac{1}{2}\psi^{(q)}(s)+\sum_{k=0}^{q-2}\binom{q-2}{k}\psi^{(q-k-1)}(s)\frac{d^k}{ds^k}(\psi(s)+\gamma)\\
&\quad\{\text{separate the first term }(k=0)\}\\
&= -\frac{1}{2}\psi^{(q)}(s)+\psi^{(q-1)}(s)(\psi(s)+\gamma)+\sum_{k=1}^{q-2}\binom{q-2}{k}\psi^{(q-k-1)}(s)\frac{d^k}{ds^k}(\psi(s)+\gamma).
\end{aligned}$$

The proof completes on writing $\frac{d^k}{ds^k}(\psi(s)+\gamma)=\psi^{(k)}(s)$.

3.1.22 $\int_0^1 \frac{\ln^{q-1}(x)\ln(1-x)}{1-x}\mathrm{d}x$

Let $q \in \mathbb{Z}_{>1}$. Then the following identity holds:

$$\int_0^1 \frac{\ln^{q-1}(x)\ln(1-x)}{1-x}\mathrm{d}x = \frac{1}{2}(-1)^q q!\,\zeta(q+1)$$
$$-\frac{1}{2}(-1)^q(q-1)!\sum_{k=1}^{q-2}\zeta(q-k)\zeta(k+1). \tag{3.41}$$

Proof. Set $s = 1$ in (3.40) then use $\psi(1) + \gamma = 0$ given in (1.147), we get

$$\int_0^1 \frac{\ln^{q-1}(x)\ln(1-x)}{1-x}\mathrm{d}x = -\frac{1}{2}\psi^{(q)}(1) + \sum_{k=1}^{q-2}\binom{q-2}{k}\psi^{(q-k-1)}(1)\psi^{(k)}(1).$$

The proof completes on using $\psi^{(a)}(1) = (-1)^{a-1}a!\,\zeta(a+1)$ given in (1.155).

Furthermore, replacing q with $2q$ in (3.41) gives

$$\int_0^1 \frac{\ln^{2q-1}(x)\ln(1-x)}{1-x}\mathrm{d}x = \frac{1}{2}(2q)!\zeta(2q+1)$$
$$-\frac{1}{2}(2q-1)!\sum_{k=1}^{2q-2}\zeta(2q-k)\zeta(k+1).$$

We simplify this sum the same way as in [31, p. 2]. By using

$$\sum_{k=1}^{2q-2} f(k) = \sum_{k=1}^{q-1} f(2k-1) + \sum_{k=1}^{q-1} f(2k),$$

we have

$$\sum_{k=1}^{2q-2}\zeta(2q-k)\zeta(k+1)$$
$$= \sum_{k=1}^{q-1}\zeta(2q-2k+1)\zeta(2k) + \sum_{k=1}^{q-1}\zeta(2q-2k)\zeta(2k+1)$$
$$\{\text{use the rule (1.2) for the second sum}\}$$
$$= \sum_{k=1}^{q-1}\zeta(2q-2k+1)\zeta(2k) + \sum_{k=1}^{q-1}\zeta(2k)\zeta(2q-2k+1)$$

$$= 2\sum_{k=1}^{q-1} \zeta(2q-2k+1)\zeta(2k). \tag{3.42}$$

Thus,

$$\begin{aligned}\int_0^1 \frac{\ln^{2q-1}(x)\ln(1-x)}{1-x}\mathrm{d}x &= \frac{1}{2}(2q)!\,\zeta(2q+1)\\ &-(2q-1)!\sum_{k=1}^{q-1}\zeta(2q-2k+1)\zeta(2k). \end{aligned}\tag{3.43}$$

Examples:

$$\int_0^1 \frac{\ln(x)\ln(1-x)}{1-x}\mathrm{d}x = \zeta(3);$$

$$\int_0^1 \frac{\ln^2(x)\ln(1-x)}{1-x}\mathrm{d}x = -\frac{1}{2}\zeta(4);$$

$$\int_0^1 \frac{\ln^3(x)\ln(1-x)}{1-x}\mathrm{d}x = 12\zeta(5)-6\zeta(2)\zeta(3);$$

$$\int_0^1 \frac{\ln^4(x)\ln(1-x)}{1-x}\mathrm{d}x = 12\zeta^2(3)-18\zeta(6).$$

The following integral is related:

$$\begin{aligned}&\int_0^1 \frac{\ln^{q-1}(x)\ln(1-x)}{x(1-x)}\mathrm{d}x\\ &= \int_0^1 \frac{\ln^{q-1}(x)\ln(1-x)}{1-x}\mathrm{d}x + \int_0^1 \frac{\ln^{q-1}(x)\ln(1-x)}{x}\mathrm{d}x\\ &\qquad\{\text{collect the results (3.41) and (3.9)}\}\\ &= \frac{1}{2}(-1)^q(q-1)!\left((q+2)\zeta(q+1)-\sum_{k=1}^{q-2}\zeta(q-k)\zeta(k+1)\right).\end{aligned}\tag{3.44}$$

Examples:

$$\int_0^1 \frac{\ln(x)\ln(1-x)}{x(1-x)}\mathrm{d}x = 2\zeta(3);$$

$$\int_0^1 \frac{\ln^2(x)\ln(1-x)}{x(1-x)}\mathrm{d}x = -\frac{5}{2}\zeta(4);$$

$$\int_0^1 \frac{\ln^3(x)\ln(1-x)}{x(1-x)}\mathrm{d}x = 18\zeta(5)-6\zeta(2)\zeta(3);$$

$$\int_0^1 \frac{\ln^4(x)\ln(1-x)}{x(1-x)}\mathrm{d}x = 12\zeta^2(3) - 42\zeta(6).$$

3.1.23 $\int_0^1 \frac{\ln^{q-1}(x)\ln(1-x^2)}{1-x^2}\mathrm{d}x$

Let $q \in \mathbb{Z}_{>1}$. Then the following identity holds:

$$\begin{aligned}\int_0^1 \frac{\ln^{q-1}(x)\ln(1-x^2)}{1-x^2}\mathrm{d}x &= (-1)^q q!\left(1-\frac{1}{2^{q+1}}\right)\zeta(q+1)\\ &-2(-1)^q(q-1)!\left(1-\frac{1}{2^q}\right)\ln(2)\zeta(q)\\ &-\frac{(-1)^q(q-1)!}{2^{q+1}}\sum_{k=1}^{q-2}(2^{q-k}-1)(2^{k+1}-1)\zeta(q-k)\zeta(k+1). \qquad (3.45)\end{aligned}$$

Proof. Set $s = \frac{1}{2}$ in (3.40) then use $\psi^{(a)}(\frac{1}{2}) + \gamma = -2\ln(2)$ given in (1.148),

$$\begin{aligned}\int_0^1 \frac{\ln^{q-1}(x)\ln(1-x)}{\sqrt{x}(1-x)}\mathrm{d}x &= -\frac{1}{2}\psi^{(q)}\left(\frac{1}{2}\right) - 2\ln(2)\psi^{(q-1)}\left(\frac{1}{2}\right)\\ &+\sum_{k=1}^{q-2}\binom{q-2}{k}\psi^{(q-k-1)}\left(\frac{1}{2}\right)\psi^{(k)}\left(\frac{1}{2}\right)\\ &\left\{\text{make use of the result of } \psi^{(a)}(1/2) \text{ given in (1.157)}\right\}\\ &= -(-1)^q q!\,(2^{-1}-2^q)\zeta(q+1) - 2(-1)^q(q-1)!\,(2^q-1)\ln(2)\zeta(q)\\ &-(-1)^q(q-2)!\sum_{k=1}^{q-2}(q-k-1)(2^{q-k}-1)(2^{k+1}-1)\zeta(q-k)\zeta(k+1).\end{aligned}$$

This sum can be further simplified. By using (1.2), we have

$$\begin{aligned}&\sum_{k=1}^{q-2}(q-k-1)(2^{q-k}-1)(2^{k+1}-1)\zeta(q-k)\zeta(k+1)\\ &= \sum_{k=1}^{q-2}k(2^{q-k}-1)(2^{k+1}-1)\zeta(q-k)\zeta(k+1)\\ &\{\text{add the sum to both sides then divide by } 2\}\\ &= \frac{1}{2}\sum_{k=1}^{q-2}(q-k+k-1)(2^{q-k}-1)(2^{k+1}-1)\zeta(q-k)\zeta(k+1)\end{aligned}$$

$$= \frac{q-1}{2} \sum_{k=1}^{q-2} (2^{q-k}-1)(2^{k+1}-1)\zeta(q-k)\zeta(k+1).$$

The proof completes on subbing $x = y^2$ in the integral:

$$\int_0^1 \frac{\ln^{q-1}(x)\ln(1-x)}{\sqrt{x}(1-x)}\mathrm{d}x = 2^q \int_0^1 \frac{\ln^{q-1}(y)\ln(1-y^2)}{1-y^2}\mathrm{d}y.$$

Examples:

$$\int_0^1 \frac{\ln(x)\ln(1-x^2)}{1-x^2}\mathrm{d}x = \frac{7}{4}\zeta(3) - \frac{3}{2}\ln(2)\zeta(2);$$

$$\int_0^1 \frac{\ln^2(x)\ln(1-x^2)}{1-x^2}\mathrm{d}x = \frac{7}{2}\ln(2)\zeta(3) - \frac{45}{16}\zeta(4);$$

$$\int_0^1 \frac{\ln^3(x)\ln(1-x^2)}{1-x^2}\mathrm{d}x = \frac{93}{4}\zeta(5) - \frac{45}{4}\ln(2)\zeta(4) - \frac{63}{8}\zeta(2)\zeta(3);$$

$$\int_0^1 \frac{\ln^4(x)\ln(1-x^2)}{1-x^2}\mathrm{d}x = \frac{93}{2}\ln(2)\zeta(5) + \frac{147}{8}\zeta^2(3) - \frac{945}{16}\zeta(6).$$

3.1.24 $\int_0^1 \frac{\ln^{q-1}(x)\ln(1+x)}{1-x}\mathrm{d}x$

Let $q \in \mathbb{Z}_{>1}$. Then the following identity holds:

$$\begin{aligned}\int_0^1 \frac{\ln^{q-1}(x)\ln(1+x)}{1-x}\mathrm{d}x &= \frac{1}{2}(-1)^q q!\,\zeta(q+1)\\ &-2(-1)^q(q-1)!\left(1-\frac{1}{2^q}\right)\ln(2)\zeta(q)\\ &-\frac{1}{2}(-1)^q(q-1)!\sum_{k=1}^{q-2}\eta(k+1)\eta(q-k).\end{aligned} \tag{3.46}$$

Proof (i). Write

$$\frac{\ln(1+x)}{1-x} = \frac{\ln(1-x^2)}{1-x^2} + \frac{x\ln(1-x^2)}{1-x^2} - \frac{\ln(1-x)}{1-x}$$

to have

$$\int_0^1 \frac{\ln^{q-1}(x)\ln(1+x)}{1-x}\mathrm{d}x = \int_0^1 \frac{\ln^{q-1}(x)\ln(1-x^2)}{1-x^2}\mathrm{d}x$$

$$+\underbrace{\int_0^1 \frac{x\ln^{q-1}(x)\ln(1-x^2)}{1-x^2}\mathrm{d}x}_{x^2\to x}-\int_0^1 \frac{\ln^{q-1}(x)\ln(1-x)}{1-x}\mathrm{d}x$$

$$=\int_0^1 \frac{\ln^{q-1}(x)\ln(1-x^2)}{1-x^2}\mathrm{d}x+(2^{-q}-1)\int_0^1 \frac{\ln^{q-1}(x)\ln(1-x)}{1-x}\mathrm{d}x.$$

These two integrals are given in (3.45) and (3.41) respectively.

The following second proof is due to Cornel Vălean and it may be found in [29]:

Proof (ii).

$$\int_0^1 \frac{\ln^{q-1}(x)\ln\left(\frac{2}{1+x}\right)}{1-x}\mathrm{d}x=\int_0^1 \frac{\ln^{q-1}(x)}{1-x}\mathrm{d}x\left(\int_x^1 \frac{\mathrm{d}y}{1+y}\right)\mathrm{d}x$$

$$\{\text{change the order of integration}\}$$

$$=\int_0^1\int_0^y \frac{\ln^{q-1}(x)}{(1+y)(1-x)}\mathrm{d}x\mathrm{d}y$$

$$\overset{x=yz}{=}\int_0^1\int_0^1 \frac{y\ln^{q-1}(yz)}{(1+y)(1-yz)}\mathrm{d}z\mathrm{d}y$$

$$\{\text{replace } z \text{ with } x \text{ then write } 1 \text{ as } 1/2+1/2\}$$

$$=\frac{1}{2}\int_0^1\int_0^1 \frac{y\ln^{q-1}(xy)}{(1+y)(1-xy)}\mathrm{d}x\mathrm{d}y+\frac{1}{2}\int_0^1\int_0^1 \frac{y\ln^{q-1}(xy)}{(1+y)(1-xy)}\mathrm{d}x\mathrm{d}y$$

$$\{\text{swap } x \text{ and } y \text{ in the second integral}\}$$

$$=\frac{1}{2}\int_0^1\int_0^1 \frac{y\ln^{q-1}(xy)}{(1+y)(1-xy)}\mathrm{d}x\mathrm{d}y+\frac{1}{2}\int_0^1\int_0^1 \frac{x\ln^{q-1}(xy)}{(1+x)(1-xy)}\mathrm{d}x\mathrm{d}y$$

$$=\frac{1}{2}\int_0^1\int_0^1 \frac{[(1+x)(1+y)-(1-xy)]\ln^{q-1}(xy)}{(1+x)(1+y)(1-xy)}\mathrm{d}x\mathrm{d}y$$

$$=\frac{1}{2}\int_0^1\int_0^1 \frac{\ln^{q-1}(xy)}{1-xy}\mathrm{d}x\mathrm{d}y-\frac{1}{2}\int_0^1\int_0^1 \frac{\ln^{q-1}(xy)}{(1+x)(1+y)}\mathrm{d}x\mathrm{d}y.$$

First integral: Let $x=\frac{t}{y}$,

$$\int_0^1\int_0^1 \frac{\ln^{q-1}(xy)}{1-xy}\mathrm{d}x\mathrm{d}y=\int_0^1\int_0^y \frac{\ln^{q-1}(t)}{y(1-t)}\mathrm{d}t\mathrm{d}y$$

$$=\int_0^1 \frac{\ln^{q-1}(t)}{1-t}\left(\int_t^1 \frac{\mathrm{d}y}{y}\right)\mathrm{d}t$$

$$=-\int_0^1 \frac{\ln^{q}(t)}{1-t}\mathrm{d}t$$

$$=-(-1)^q q!\,\zeta(q+1).$$

Second integral:

$$\int_0^1\int_0^1 \frac{\ln^{q-1}(xy)}{(1+x)(1+y)}\mathrm{d}x\mathrm{d}y = \int_0^1\int_0^1 \frac{[\ln(x)+\ln(y)]^{q-1}}{(1+x)(1+y)}\mathrm{d}x\mathrm{d}y$$

$$\left\{\text{use the binomial theorem } (x+y)^a = \sum_{k=0}^{a}\binom{a}{k} y^{a-k}\,x^k\right\}$$

$$= \int_0^1\int_0^1 \frac{\sum_{k=0}^{q-1}\binom{q-1}{k}\ln^{q-k-1}(y)\ln^k(x)}{(1+x)(1+y)}\mathrm{d}x\mathrm{d}y$$

$$= \sum_{k=0}^{q-1}\binom{q-1}{k}\left(\int_0^1 \frac{\ln^{q-k-1}(y)}{1+y}\mathrm{d}y\right)\left(\int_0^1 \frac{\ln^k(x)}{1+x}\mathrm{d}x\right)$$

$$\{\text{separate the first and last terms}\}$$

$$= \underbrace{\binom{q-1}{0}\left(\int_0^1 \frac{\ln^{q-1}(y)}{1+y}\mathrm{d}y\right)\left(\int_0^1 \frac{\mathrm{d}x}{1+x}\right)}_{\text{first term}}$$

$$+ \underbrace{\binom{q-1}{q-1}\left(\int_0^1 \frac{\mathrm{d}y}{1+y}\right)\left(\int_0^1 \frac{\ln^{q-1}(x)}{1+x}\mathrm{d}x\right)}_{\text{last term}}$$

$$+ \sum_{k=1}^{q-2}\binom{q-1}{k}\left(\int_0^1 \frac{\ln^{q-k-1}(y)}{1+x}\mathrm{d}y\right)\left(\int_0^1 \frac{\ln^k(x)}{1+x}\mathrm{d}x\right)$$

$$\{\text{make use of the definition } (1.53)\}$$

$$= -2(-1)^q(q-1)!\,\ln(2)\eta(q)$$

$$-(-1)^q\sum_{k=1}^{q-2}\binom{q-1}{k}k!\,(q-k-1)!\,\eta(k+1)\eta(q-k)$$

$$= -2(-1)^q(q-1)!\,\ln(2)\eta(q) - (-1)^q(q-1)!\sum_{k=1}^{q-2}\eta(k+1)\eta(q-k).$$

Combine the two integrals,

$$\int_0^1 \frac{\ln^{q-1}(x)\ln\left(\frac{2}{1+x}\right)}{1-x}\mathrm{d}x = -\frac{1}{2}(-1)^q q!\,\zeta(q+1) + (-1)^q(q-1)!\,\ln(2)\eta(q)$$

$$+\frac{1}{2}(-1)^q(q-1)!\sum_{k=1}^{q-2}\eta(q-k)\eta(k+1). \tag{3.47}$$

The proof completes on writing

$$\int_0^1 \frac{\ln^{q-1}(x)\ln\left(\frac{2}{1+x}\right)}{1-x}\mathrm{d}x = \ln(2)\int_0^1 \frac{\ln^{q-1}(x)}{1-x}\mathrm{d}x - \int_0^1 \frac{\ln^{q-1}(x)\ln(1+x)}{1-x}\mathrm{d}x$$
$$= -(-1)^q(q-1)!\ln(2)\zeta(q) - \int_0^1 \frac{\ln^{q-1}(x)\ln(1+x)}{1-x}\mathrm{d}x.$$

Examples:

$$\int_0^1 \frac{\ln(x)\ln(1+x)}{1-x}\mathrm{d}x = \zeta(3) - \frac{3}{2}\ln(2)\zeta(2);$$
$$\int_0^1 \frac{\ln^2(x)\ln(1+x)}{1-x}\mathrm{d}x = -\frac{19}{8}\zeta(4) + \frac{7}{2}\ln(2)\zeta(3);$$
$$\int_0^1 \frac{\ln^3(x)\ln(1+x)}{1-x}\mathrm{d}x = 12\zeta(5) - \frac{45}{4}\ln(2)\zeta(4) - \frac{9}{4}\zeta(2)\zeta(3);$$
$$\int_0^1 \frac{\ln^4(x)\ln(1+x)}{1-x}\mathrm{d}x = -\frac{333}{8}\zeta(6) + \frac{93}{2}\ln(2)\zeta(5) + \frac{27}{4}\zeta^2(3).$$

The following integral is related:

$$\int_0^1 \frac{\ln^{q-1}(x)\ln(1+x)}{x(1-x)}\mathrm{d}x$$
$$= \int_0^1 \frac{\ln^{q-1}(x)\ln(1+x)}{x}\mathrm{d}x + \int_0^1 \frac{\ln^{q-1}(x)\ln(1+x)}{1-x}\mathrm{d}x$$
$$\{\text{group the results (3.12) and (3.46) }\}$$
$$= (-1)^{q-1}(q-1)!\left[2\left(1-\frac{1}{2^q}\right)\ln(2)\zeta(q) + \left(1-\frac{1}{2^q}-\frac{q}{2}\right)\zeta(q+1)\right.$$
$$\left.+\frac{1}{2}\sum_{k=1}^{q-2}\eta(q-k)\eta(k+1)\right]. \tag{3.48}$$

Let's replace q with $2q$ in (3.46),

$$\int_0^1 \frac{\ln^{2q-1}(x)\ln(1+x)}{1-x}\mathrm{d}x = \frac{1}{2}(2q)!\,\zeta(2q+1)$$
$$-(2q-1)!\,(1-2^{-2q})\ln(2)\eta(2q) - \frac{1}{2}(2q-1)!\sum_{k=1}^{2q-2}\eta(k+1)\eta(2q-k).$$

We simplify this sum the same way as in (3.42). By using

$$\sum_{k=1}^{2q-2} f(k) = \sum_{k=1}^{q-1} f(2k-1) + \sum_{k=1}^{q-1} f(2k),$$

we have

$$\begin{aligned}
&\sum_{k=1}^{2q-2} \eta(k+1)\eta(2q-k) \\
&= \sum_{k=1}^{q-1} \eta(2k)\eta(2q-2k+1) + \sum_{k=1}^{q-1} \eta(2k+1)\eta(2q-2k) \\
&\qquad \{\text{use the rule (1.2) for the second sum}\} \\
&= \sum_{k=1}^{q-1} \eta(2k)\eta(2q-2k+1) + \sum_{k=1}^{q-1} \eta(2q-2k+1)\eta(2k) \\
&= 2\sum_{k=1}^{q-1} \eta(2k)\eta(2q-2k+1). \qquad (3.49)
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_0^1 \frac{\ln^{2q-1}(x)\ln(1+x)}{1-x}\mathrm{d}x \\
&= \frac{1}{2}(2q)!\,\zeta(2q+1) - 2(2q-1)!(1-2^{-2q})\ln(2)\zeta(2q) \\
&\quad -(2q-1)!\sum_{k=1}^{q-1} \eta(2k)\eta(2q-2k+1). \qquad (3.50)
\end{aligned}$$

For a short and different method to derive (3.50), divide both sides of (1.101):

$$\int_0^1 \frac{(1+2xy+x^2)\ln^{2q-1}(y)}{(1+xy)(x+y)}\mathrm{d}y = -2(2q-1)!\sum_{k=0}^{q} \frac{\eta(2q-2k)}{(2k)!}\ln^{2k}(x)$$

by $1+x$ then integrate using $\int_0^1 \frac{\ln^a(x)}{1+x}\mathrm{d}x = (-1)^a a!\,\eta(a+1)$, we have

$$\begin{aligned}
&-2(2q-1)!\sum_{k=0}^{q} \eta(2q-2k)\eta(2k+1) \\
&= \int_0^1\int_0^1 \frac{(1+2xy+x^2)\ln^{2q-1}(y)}{(1+x)(1+xy)(x+y)}\mathrm{d}y\mathrm{d}x \\
&\qquad \{\text{change the order of integration}\}
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \ln^{2q-1}(y)\left(\int_0^1 \frac{1+2xy+x^2}{(1+xy)(x+y)(1+x)}\mathrm{d}x\right)\mathrm{d}y\\
&= \int_0^1 \ln^{2q-1}(y)\left(\frac{2\ln(1+y)}{1-y}+\frac{\ln(1+y)}{y}-\frac{\ln(y)}{1-y}-\frac{2\ln(2)}{1-y}\right)\mathrm{d}y\\
&\qquad \{\text{integrate the second integral by parts}\}\\
&= 2\int_0^1 \frac{\ln^{2q-1}(y)\ln(1+y)}{1-y}\mathrm{d}y-\frac{1}{2q}\int_0^1\frac{\ln^{2q}(y)}{1+y}\mathrm{d}y\\
&\quad -\int_0^1\frac{\ln^{2q}(y)}{1-y}\mathrm{d}y-2\ln(2)\int_0^1\frac{\ln^{2q-1}(y)}{1-y}\mathrm{d}y\\
&= 2\int_0^1 \frac{\ln^{2q-1}(y)\ln(1+y)}{1-y}\mathrm{d}y-(2q-1)!\,\eta(2q+1)\\
&\quad -(2q)!\,\zeta(2q+1)+2(2q-1)!\ln(2)\zeta(2q).
\end{aligned}
$$

For the sum, separate the first and last terms,

$$
\begin{aligned}
&\sum_{k=0}^{q}\eta(2q-2k)\eta(2k+1)\\
&= \eta(2q)\eta(1)+\eta(2q+1)\eta(0)+\sum_{k=1}^{q-1}\eta(2q-2k)\eta(2k+1)\\
&\qquad \{\text{use the rule (1.2) for the sum}\}\\
&= \ln(2)\eta(2q)+\frac{1}{2}\eta(2q+1)+\sum_{k=1}^{q-1}\eta(2k)\eta(2q-2k+1).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\int_0^1 \frac{\ln^{2q-1}(y)\ln(1+y)}{1-y}\mathrm{d}y\\
&= \frac{1}{2}(2q)!\,\zeta(2q+1)-2(2q-1)!(1-2^{-2q})\ln(2)\zeta(2q)\\
&\quad -(2q-1)!\sum_{k=1}^{q-1}\eta(2k)\eta(2q-2k+1),
\end{aligned}
$$

which matches (3.50).

3.1.25 $\int_0^1 \frac{\ln^{2q-1}(x)\ln(1+x)}{1-x^2}\mathrm{d}x$

Let $q \in \mathbb{Z}_{>1}$. Then the following identity holds:

$$\int_0^1 \frac{\ln^{2q-1}(x)\ln(1+x)}{1-x^2}\mathrm{d}x = -(2q-1)!\left(1-\frac{1}{2^{2q}}\right)\ln(2)\zeta(2q) \\ +\frac{1}{2}(2q-1)(2q-1)!\left(1-\frac{1}{2^{2q+1}}\right)\zeta(2q+1) \\ -(2q-1)!\sum_{k=1}^{q-1}(1-2^{2k-2q-1})\eta(2k)\zeta(2q-2k+1). \tag{3.51}$$

Proof.

$$\int_0^1 \frac{\ln^{2q-1}(x)\ln(1+x)}{1-x^2}\mathrm{d}x \\ =\frac{1}{2}\int_0^1 \frac{\ln^{2q-1}(x)\ln(1+x)}{1+x}\mathrm{d}x+\frac{1}{2}\int_0^1 \frac{\ln^{2q-1}(x)\ln(1+x)}{1-x}\mathrm{d}x.$$

These two integrals are given in (3.38) and (3.50) respectively.

Examples:

$$\int_0^1 \frac{\ln(x)\ln(1+x)}{1-x^2}\mathrm{d}x = \frac{7}{16}\zeta(3)-\frac{3}{4}\ln(2)\zeta(2);$$

$$\int_0^1 \frac{\ln^3(x)\ln(1+x)}{1-x^2}\mathrm{d}x = \frac{279}{32}\zeta(5)-\frac{45}{8}\ln(2)\zeta(4)-\frac{21}{8}\zeta(2)\zeta(3);$$

$$\int_0^1 \frac{\ln^5(x)\ln(1+x)}{1-x^2}\mathrm{d}x = \frac{9525}{32}\zeta(7)-\frac{945}{8}\ln(2)\zeta(6)-\frac{465}{8}\zeta(2)\zeta(5) \\ -\frac{735}{8}\zeta(3)\zeta(4);$$

$$\int_0^1 \frac{\ln^7(x)\ln(1+x)}{1-x^2}\mathrm{d}x = \frac{1126755}{64}\zeta(9)-\frac{80325}{16}\ln(2)\zeta(8)-\frac{40005}{16}\zeta(2)\zeta(7) \\ -\frac{68355}{16}\zeta(3)\zeta(6)-\frac{68355}{16}\zeta(4)\zeta(5).$$

3.1.26 $\int_0^1 \frac{\ln^{2q-2}(x)\ln(1-x)\ln(1+x)}{x}\mathrm{d}x$

Let $q \in \mathbb{Z}^+$. Then the following identity holds:

$$\int_0^1 \frac{\ln^{2q-2}(x)\ln(1-x)\ln(1+x)}{x}\mathrm{d}x = \left(\frac{2q+1}{2^{2q+1}}-1\right)(2q-2)!\,\zeta(2q+1)$$
$$+(2q-2)!\sum_{k=1}^{q-1}(2^{1-2k}-2^{1-2q})\zeta(2k)\zeta(2q-2k+1). \tag{3.52}$$

Proof. Write

$$\ln(1-x)\ln(1+x) = \frac{1}{2}\ln^2(1-x^2) - \frac{1}{2}\ln^2(1-x) - \frac{1}{2}\ln^2(1+x)$$

to have

$$\int_0^1 \frac{\ln^{2q-2}(x)\ln(1-x)\ln(1+x)}{x}\mathrm{d}x = \frac{1}{2}\underbrace{\int_0^1 \frac{\ln^{2q-2}(x)\ln^2(1-x^2)}{x}\mathrm{d}x}_{x^2\to x}$$
$$-\frac{1}{2}\int_0^1 \frac{\ln^{2q-2}(x)\ln^2(1-x)}{x}\mathrm{d}x - \frac{1}{2}\int_0^1 \frac{\ln^{2q-2}(x)\ln^2(1+x)}{x}\mathrm{d}x$$
$$= \frac{2^{1-2q}-1}{2}\int_0^1 \frac{\ln^{2q-2}(x)\ln^2(1-x)}{x}\mathrm{d}x - \frac{1}{2}\int_0^1 \frac{\ln^{2q-2}(x)\ln^2(1+x)}{x}\mathrm{d}x$$
$$\{\text{integrate both integrals by parts}\}$$
$$= \frac{2^{1-2q}-1}{2q-1}\int_0^1 \frac{\ln^{2q-1}(x)\ln(1-x)}{1-x}\mathrm{d}x + \frac{1}{2q-1}\int_0^1 \frac{\ln^{2q-1}(x)\ln(1+x)}{1+x}\mathrm{d}x.$$

These two integrals are given in (3.43) and (3.38) respectively.

A different proof may be found in [31].

Examples:

$$\int_0^1 \frac{\ln(1-x)\ln(1+x)}{x}\mathrm{d}x = -\frac{5}{8}\zeta(3);$$
$$\int_0^1 \frac{\ln^2(x)\ln(1-x)\ln(1+x)}{x}\mathrm{d}x = \frac{3}{4}\zeta(2)\zeta(3) - \frac{27}{16}\zeta(5);$$
$$\int_0^1 \frac{\ln^4(x)\ln(1-x)\ln(1+x)}{x}\mathrm{d}x = \frac{9}{4}\zeta(3)\zeta(4) + \frac{45}{4}\zeta(2)\zeta(5) - \frac{363}{16}\zeta(7);$$
$$\int_0^1 \frac{\ln^6(x)\ln(1-x)\ln(1+x)}{x}\mathrm{d}x = \frac{135}{8}\zeta(3)\zeta(6) + \frac{675}{8}\zeta(4)\zeta(5)$$

$$+\frac{2835}{8}\zeta(2)\zeta(7)-\frac{22635}{32}\zeta(9).$$

3.1.27 $\int_0^1 \frac{\ln^{2q-1}(x)\ln(1-x)}{1+x}\mathrm{d}x$

Let $q\in\mathbb{Z}^+$. Then the following identity holds:

$$\begin{aligned}\int_0^1 \frac{\ln^{2q-1}(x)\ln(1-x)}{1+x}\mathrm{d}x &= 2(2q-1)!\left(\frac{1}{2^{2q}}-1\right)\ln(2)\zeta(2q)\\ &+(2q-1)!\left(q+1-\frac{2q+1}{2^{2q+1}}\right)\zeta(2q+1)\\ &-(2q-1)!\sum_{k=1}^{q-1}\zeta(2k)\eta(2q-2k+1).\end{aligned}\tag{3.53}$$

Proof. Force integration by parts,

$$\begin{aligned}&\int_0^1 \frac{\ln^{2q-1}(x)\ln(1-x)}{1+x}\mathrm{d}x\\ &=\int_0^1 \frac{\ln^{2q-1}(x)\ln(1+x)}{1-x}\mathrm{d}x-(2q-1)\int_0^1 \frac{\ln^{2q-2}(x)\ln(1-x)\ln(1+x)}{x}\mathrm{d}x.\end{aligned}$$

These two integrals are given in (3.50) and (3.52) respectively.

Examples:

$$\int_0^1 \frac{\ln(x)\ln(1-x)}{1+x}\mathrm{d}x=\frac{13}{8}\zeta(3)-\frac{3}{2}\ln(2)\zeta(2);$$

$$\int_0^1 \frac{\ln^3(x)\ln(1-x)}{1+x}\mathrm{d}x=\frac{273}{16}\zeta(5)-\frac{9}{2}\zeta(2)\zeta(3)-\frac{45}{4}\ln(2)\zeta(4);$$

$$\begin{aligned}\int_0^1 \frac{\ln^5(x)\ln(1-x)}{1+x}\mathrm{d}x&=\frac{7575}{16}\zeta(7)-\frac{225}{2}\zeta(2)\zeta(5)-90\zeta(3)\zeta(4)\\ &-\frac{945}{4}\ln(2)\zeta(6);\end{aligned}$$

$$\begin{aligned}\int_0^1 \frac{\ln^7(x)\ln(1-x)}{1+x}\mathrm{d}x&=\frac{803565}{32}\zeta(9)-\frac{19845}{4}\zeta(2)\zeta(7)-4725\zeta(4)\zeta(5)\\ &-3780\zeta(3)\zeta(6)-\frac{80325}{8}\ln(2)\zeta(8).\end{aligned}$$

The following integral is related:

$$\begin{aligned}
&\int_0^1 \frac{\ln^{2q-1}(x)\ln(1-x)}{x(1+x)}\mathrm{d}x\\
&= \int_0^1 \frac{\ln^{2q-1}(x)\ln(1-x)}{x}\mathrm{d}x - \int_0^1 \frac{\ln^{2q-1}(x)\ln(1-x)}{1+x}\mathrm{d}x\\
&\qquad\{\text{group the results (3.9) and (3.53)}\}\\
&= -(2q-1)!\left[\left(q-\frac{2q+1}{2^{2q+1}}\right)\zeta(2q+1) - 2\left(1-\frac{1}{2^{2q}}\right)\ln(2)\zeta(2q)\right.\\
&\qquad\left. - \sum_{k=1}^{q-1}\zeta(2k)\eta(2q-2k+1)\right]. \qquad (3.54)
\end{aligned}$$

3.1.28 $\int_0^1 \frac{\ln^{2q}(x)\ln(1+x)}{1+x^2}\mathrm{d}x$

Let $q \in \mathbb{Z}_{\geq 0}$. Then the following identity holds:

$$\begin{aligned}
\int_0^1 \frac{\ln^{2q}(x)\ln(1+x)}{1+x^2}\mathrm{d}x &= \frac{(2q)!}{2}\ln(2)\beta(2q+1) - \frac{(2q+1)!}{2}\beta(2q+2)\\
&\quad + (2q)!\sum_{k=0}^{q}\eta(2q-2k)\beta(2k+2). \qquad (3.55)
\end{aligned}$$

Proof. Divide both sides of (1.100):

$$-2(2q)!\sum_{k=0}^{q}\frac{\eta(2q-2k)}{(2k+1)!}\ln^{2k+1}(x) = \int_0^1 \frac{(1-x^2)\ln^{2q}(y)}{(1+xy)(x+y)}\mathrm{d}y$$

by $1+x^2$ then integrate from $x=0\to 1$ using $\int_0^1 \frac{\ln^a(x)}{1+x^2}\mathrm{d}x = (-1)^a a!\,\beta(a+1)$, which is given in (1.62), we get

$$\begin{aligned}
2(2q)!\sum_{k=0}^{q}\eta(2q-2k)\beta(2k+2) &= \int_0^1\int_0^1 \frac{(1-x^2)\ln^{2q}(y)}{(1+x^2)(1+xy)(x+y)}\mathrm{d}y\mathrm{d}x\\
&\qquad\{\text{change the order of integration}\}\\
&= \int_0^1 \ln^{2q}(y)\left(\int_0^1 \frac{1-x^2}{(1+x^2)(1+xy)(x+y)}\mathrm{d}x\right)\mathrm{d}y\\
&= \int_0^1 \ln^{2q}(y)\left(\frac{2\ln(1+y)-\ln(y)-\ln(2)}{1+y^2}\right)\mathrm{d}y
\end{aligned}$$

$$= 2\int_0^1 \frac{\ln^{2q}(y)\ln(1+y)}{1+y^2}\mathrm{d}y - \int_0^1 \frac{\ln^{2q+1}(y)}{1+y^2}\mathrm{d}y - \ln(2)\int_0^1 \frac{\ln^{2q}(y)}{1+y^2}\mathrm{d}y$$

$$= 2\int_0^1 \frac{\ln^{2q}(x)\ln(1+x)}{1+x^2}\mathrm{d}x + (2q+1)!\,\beta(2q+2) - (2q)!\ln(2)\beta(2q+1).$$

Examples:

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}\mathrm{d}x = \frac{\pi}{4}\ln(2);$$

$$\int_0^1 \frac{\ln^2(x)\ln(1+x)}{1+x^2}\mathrm{d}x = G\zeta(2) + \ln(2)\beta(3) - 2\beta(4);$$

$$\int_0^1 \frac{\ln^4(x)\ln(1+x)}{1+x^2}\mathrm{d}x = 21G\zeta(4) + 12\zeta(2)\beta(4) + 12\ln(2)\beta(5) - 48\beta(6);$$

$$\int_0^1 \frac{\ln^6(x)\ln(1+x)}{1+x^2}\mathrm{d}x = \frac{1395}{2}G\zeta(6) + 630\zeta(4)\beta(4) + 360\zeta(2)\beta(6)$$
$$+360\ln(2)\beta(7) - 2160\beta(8).$$

The integral $\int_0^1 \frac{\ln(1+x)}{1+x^2}\mathrm{d}x$ is also calculated in (3.96).
The following is a related integral:

$$\int_0^\infty \frac{\ln^{2q}(x)\ln(1+x)}{1+x^2}\mathrm{d}x$$

$$= \int_0^1 \frac{\ln^{2q}(x)\ln(1+x)}{1+x^2}\mathrm{d}x + \underbrace{\int_1^\infty \frac{\ln^{2q}(x)\ln(1+x)}{1+x^2}\mathrm{d}x}_{x\to 1/x}$$

$$= 2\int_0^1 \frac{\ln^{2q}(x)\ln(1+x)}{1+x^2}\mathrm{d}x - \int_0^1 \frac{\ln^{2q+1}(x)}{1+x^2}\mathrm{d}x$$

$$\{\text{recall the results (3.55) and (1.62)}\}$$

$$= (2q)!\ln(2)\beta(2q+1) + 2(2q)!\sum_{k=0}^{q}\eta(2q-2k)\beta(2k+2). \tag{3.56}$$

3.1.29 $\int_0^1 \frac{\ln^{2q}(x)\ln(1-x)}{1+x^2}\mathrm{d}x$

Let $q \in \mathbb{Z}_{\geq 0}$. Then the following identity holds:

$$\int_0^1 \frac{\ln^{2q}(x)\ln(1-x)}{1+x^2}\mathrm{d}x = \frac{(2q)!}{2}\ln(2)\beta(2q+1) - \frac{(2q+1)!}{2}\beta(2q+2)$$

(continued)

$$+(2q)!\sum_{k=0}^{q}\zeta(2q-2k)\beta(2k+2). \tag{3.57}$$

Proof. Consider the real parts of (1.111),

$$-2(2q)!\sum_{k=0}^{q}\frac{\zeta(2q-2k)}{(2k+1)!}\ln^{2k+1}(x)=\Re\int_0^1\frac{(1-x^2)\ln^{2q}(y)}{(1-xy)(x-y)}\mathrm{d}y.$$

Divide both sides by $1+x^2$ then integrate from $x=0\to 1$,

$$\begin{aligned}2(2q)!\sum_{k=0}^{q}\zeta(2q-2k)\beta(2k+2)&=\Re\int_0^1\int_0^1\frac{(1-x^2)\ln^{2q}(y)}{(1+x^2)(1-yx)(x-y)}\mathrm{d}y\mathrm{d}x\\&=\int_0^1\ln^{2q}(y)\left[\Re\int_0^1\frac{1-x^2}{(1+x^2)(1-yx)(x-y)}\mathrm{d}x\right]\mathrm{d}y\\&=\int_0^1\ln^{2q}(y)\left[\Re\left(\frac{2\ln(1-y)-\ln(-2y)}{1+y^2}\right)\right]\mathrm{d}y\\&=\int_0^1\ln^{2q}(y)\left[\Re\left(\frac{2\ln(1-y)-\ln(y)-\ln(2)-i\,\pi}{1+y^2}\right)\right]\mathrm{d}y\\&=2\int_0^1\frac{\ln^{2q}(y)\ln(1-y)}{1+y^2}\mathrm{d}y-\int_0^1\frac{\ln^{2q+1}(y)}{1+y^2}\mathrm{d}y-\int_0^1\frac{\ln(2)\ln^{2q}(y)}{1+y^2}\mathrm{d}y\\&=2\int_0^1\frac{\ln^{2q}(y)\ln(1-y)}{1+y^2}\mathrm{d}y+(2q+1)!\beta(2q+2)-(2)!\ln(2)\beta(2q+1).\end{aligned}$$

Examples:

$$\int_0^1\frac{\ln(1-x)}{1+x^2}\mathrm{d}x=\frac{\pi}{8}\ln(2)-G;$$

$$\int_0^1\frac{\ln^2(x)\ln(1-x)}{1+x^2}\mathrm{d}x=2G\zeta(2)+\ln(2)\beta(3)-4\beta(4);$$

$$\int_0^1\frac{\ln^4(x)\ln(1-x)}{1+x^2}\mathrm{d}x=24G\zeta(4)+24\zeta(2)\beta(4)+12\ln(2)\beta(5)-72\beta(6);$$

$$\begin{aligned}\int_0^1\frac{\ln^6(x)\ln(1-x)}{1+x^2}\mathrm{d}x&=720G\zeta(6)+720\zeta(4)\beta(4)+720\zeta(2)\beta(6)\\&\quad+360\ln(2)\beta(7)-2880\beta(8).\end{aligned}$$

The integral $\int_0^1\frac{\ln(1-x)}{1+x^2}\mathrm{d}x$ is also calculated in (3.97).

3.1.30 $\int_0^1 \frac{\ln^{2q}(x)\ln(1-x^2)}{1+x^2}\mathrm{d}x$

Let $q \in \mathbb{Z}_{\geq 0}$. Then the following identity holds:

$$\int_0^1 \frac{\ln^{2q}(x)\ln(1-x^2)}{1+x^2}\mathrm{d}x = (2q)!\ln(2)\beta(2q+1) - (2q+1)!\,\beta(2q+2)$$
$$+2(2q)!\sum_{k=0}^{q}(1-2^{2k-2q})\zeta(2q-2k)\beta(2k+2). \tag{3.58}$$

Proof. By writing $\ln(1-x^2) = \ln(1-x) + \ln(1+x)$, we have

$$\int_0^1 \frac{\ln^{2q}(x)\ln(1-x^2)}{1+x^2}\mathrm{d}x$$
$$= \int_0^1 \frac{\ln^{2q}(x)\ln(1-x)}{1+x^2}\mathrm{d}x + \int_0^1 \frac{\ln^{2q}(x)\ln(1+x)}{1+x^2}\mathrm{d}x.$$

These two integral are given in (3.57) and (3.55) respectively.

3.1.31 $\int_0^1 \frac{\ln^{2q}(x)\ln\left(\frac{1-x}{1+x}\right)}{1+x^2}\mathrm{d}x$

Let $q \in \mathbb{Z}_{\geq 0}$. Then the following identity holds:

$$\int_0^1 \frac{\ln^{2q}(x)\ln\left(\frac{1-x}{1+x}\right)}{1+x^2}\mathrm{d}x = \frac{(2q)!}{2^{2q-1}}\sum_{k=0}^{q}2^{2k}\zeta(2q-2k)\beta(2k+2). \tag{3.59}$$

Proof. Set $z = x^2$ in (1.111) then consider the real parts,

$$-4(2q)!\sum_{k=0}^{q}2^{2k}\frac{\zeta(2q-2k)}{(2k+1)!}\ln^{2k+1}(x) = \Re\int_0^1 \frac{(1-x^4)\ln^{2q}(y)}{(1-yx^2)(x^2-y)}\mathrm{d}y.$$

Divide both sides by $1+x^2$ then integrate from $x = 0 \to 1$,

$$4(2q)!\sum_{k=0}^{q}2^{2k}\zeta(2q-2k)\beta(2k+2) = \Re\int_0^1\int_0^1 \frac{(1-x^2)\ln^{2q}(y)}{(1-yx^2)(x^2-y)}\mathrm{d}y\mathrm{d}x$$
$$= \int_0^1 \ln^{2q}(y)\left[\Re\int_0^1 \frac{1-x^2}{(1-yx^2)(x^2-y)}\mathrm{d}x\right]\mathrm{d}y$$

$$
\begin{aligned}
&= \int_0^1 \frac{\ln^{2q}(y)}{1+y}\left[\Re\left(-\frac{\operatorname{arctanh}\sqrt{y}+\operatorname{arccoth}\sqrt{y}}{\sqrt{y}}\right)\right]\mathrm{d}y\\
&= \int_0^1 \frac{\ln^{2q}(y)}{1+y}\left[\Re\left(\frac{\ln\left(\frac{1-\sqrt{y}}{1+\sqrt{y}}\right)}{\sqrt{y}}+\frac{i\,\pi}{2\sqrt{y}}\right)\right]\mathrm{d}y\\
&= \int_0^1 \frac{\ln^{2q}(y)\ln\left(\frac{1-\sqrt{y}}{1+\sqrt{y}}\right)}{(1+y)\sqrt{y}}\mathrm{d}y\\
&\overset{\sqrt{y}=x}{=} 2^{2q+1}\int_0^1 \frac{\ln^{2q}(x)\ln\left(\frac{1-x}{1+x}\right)}{1+x^2}\mathrm{d}x.
\end{aligned}
$$

A different proof is by taking the difference of (3.55) and (3.57).

3.1.32 $\int_0^1 \frac{\ln^{2q}(x)\arctan(x)}{1-x^2}\mathrm{d}x$

Let $q \in \mathbb{Z}^+$. Then the following identity holds:

$$
\begin{aligned}
\int_0^1 \frac{\ln^{2q}(x)\arctan(x)}{1-x^2}\mathrm{d}x &= (2q)!\left(1-\frac{1}{2^{2q+1}}\right)\frac{\pi}{4}\zeta(2q+1)\\
&-\frac{(2q)!}{4^q}\sum_{k=0}^{q}2^{2k}\eta(2q-2k)\beta(2k+2). \qquad (3.60)
\end{aligned}
$$

Proof. Set $z=x^2$ in (1.100), we have

$$
-4(2q)!\sum_{k=0}^{q}2^{2k}\frac{\eta(2q-2k)}{(2k+1)!}\ln^{2k+1}(x) = \int_0^1 \frac{(1-x^4)\ln^{2q}(y)}{(1+x^2y)(x^2+y)}\mathrm{d}y.
$$

Divide both sides by $1+x^2$ then integrate from $x=0\to 1$,

$$
\begin{aligned}
4(2q)!\sum_{k=0}^{q}2^{2k}\eta(2q-2k)\beta(2k+2) &= \int_0^1\int_0^1 \frac{(1-x^2)\ln^{2q}(y)}{(1+x^2y)(x^2+y)}\mathrm{d}y\mathrm{d}x\\
&= \int_0^1 \ln^{2q}(y)\left(\int_0^1 \frac{1-x^2}{(1+x^2y)(x^2+y)}\mathrm{d}x\right)\mathrm{d}y\\
&= \int_0^1 \ln^{2q}(y)\left(\frac{\arctan(1/\sqrt{y})-\arctan(\sqrt{y})}{\sqrt{y}(1-y)}\right)\mathrm{d}y\\
&\overset{\sqrt{y}=x}{=} 2^{2q+1}\int_0^1 \ln^{2q}(x)\left(\frac{\arctan(1/x)-\arctan(x)}{1-x^2}\right)\mathrm{d}x
\end{aligned}
$$

$$\{\text{use } \arctan(1/x) = \pi/2 - \arctan(x) \text{ for } x > 0\}$$

$$= 2^{2q}\pi \int_0^1 \frac{\ln^{2q}(x)}{1-x^2}\mathrm{d}x - 2^{2q+2}\int_0^1 \frac{\ln^{2q}(x)\arctan(x)}{1-x^2}\mathrm{d}x. \tag{3.61}$$

Employ (3.14) for the first integral and the proof is finalized.

Examples:

$$\int_0^1 \frac{\ln^2(x)\arctan(x)}{1-x^2}\mathrm{d}x = \frac{7\pi}{16}\zeta(3) - \frac{1}{4}G\zeta(2) - \beta(4);$$

$$\int_0^1 \frac{\ln^4(x)\arctan(x)}{1-x^2}\mathrm{d}x = \frac{93\pi}{16}\zeta(5) - \frac{21}{16}G\zeta(4) - 3\zeta(2)\beta(4) - 12\beta(6);$$

$$\int_0^1 \frac{\ln^6(x)\arctan(x)}{1-x^2}\mathrm{d}x = \frac{5715\pi}{32}\zeta(7) - \frac{1395}{128}G\zeta(6) - \frac{315}{8}\zeta(4)\beta(4)$$
$$-90\zeta(2)\beta(6) - 360\beta(8);$$

$$\int_0^1 \frac{\ln^8(x)\arctan(x)}{1-x^2}\mathrm{d}x = \frac{160965\pi}{16}\zeta(9) - \frac{40005}{256}G\zeta(8) - \frac{9765}{16}\zeta(6)\beta(4)$$
$$-2205\zeta(4)\beta(6) - 5040\zeta(2)\beta(8) - 20160\beta(10).$$

If we let $x = \frac{1-y}{1+y}$ then write $\arctan\left(\frac{1-y}{1+y}\right) = \frac{\pi}{4} - \arctan(y)$, we get

$$\int_0^1 \frac{\ln^{2q}\left(\frac{1-x}{1+x}\right)\arctan(x)}{x}\mathrm{d}x = 2\int_0^1 \frac{\ln^{2q}(y)(\frac{\pi}{4} - \arctan(y))}{1-y^2}\mathrm{d}y$$

$$= \frac{\pi}{2}\int_0^1 \frac{\ln^{2q}(y)}{1-y^2}\mathrm{d}y - 2\int_0^1 \frac{\ln^{2q}(y)\arctan(y)}{1-y^2}\mathrm{d}y$$

$$\{\text{recall the relation (3.61)}\}$$

$$= \frac{(2q)!}{2^{2q-1}}\sum_{k=0}^{q} 2^{2k}\eta(2q-2k)\beta(2k+2). \tag{3.62}$$

3.1.33 $\int_0^1 \frac{\ln^{q-1}(x)\ln(1+x^2)}{1-x^2}\mathrm{d}x$

Let $q \in \mathbb{Z}_{>1}$. Then the following identity holds:

$$\int_0^1 \frac{\ln^{q-1}(x)\ln(1+x^2)}{1-x^2}\mathrm{d}x = (-1)^q(q-1)!\left(\frac{1}{2^q} - 1\right)\ln(2)\zeta(q)$$
$$+(-1)^q q!\left(1 - \frac{1}{2^{q+1}}\right)\zeta(q+1)$$

(continued)

$$-(-1)^q(q-1)!\sum_{k=0}^{q-1}\beta(q-k)\beta(k+1). \tag{3.63}$$

The following proof is due to Sean Stewart and it may be found in [23, p. 92 – 95]:
Proof. Using the special case of the Leibniz integral rule (see [48]):

$$\frac{d}{dx}\int_a^{f(x)} g(t)\mathrm{d}t = g(f(x))\cdot\frac{d}{dx}f(x),$$

where a is a constant, we can write

$$\frac{d}{dx}\int_0^x \frac{\ln^{q-1}(t)}{1-t^2}\mathrm{d}t = \frac{\ln^{q-1}(x)}{1-x^2}.$$

Multiply both sides by $\ln(1+x^2)$ then integrate from $x=0\to 1$,

$$\int_0^1 \frac{\ln^{q-1}(x)\ln(1+x^2)}{1-x^2}\mathrm{d}x = \int_0^1 \ln(1+x^2)\left(\frac{d}{dx}\int_0^x \frac{\ln^{q-1}(t)}{1-t^2}\mathrm{d}t\right)\mathrm{d}x$$

$$\overset{\text{IBP}}{=} \ln(1+x^2)\int_0^x \frac{\ln^{q-1}(t)}{1-t^2}\mathrm{d}t\bigg|_{x=0}^{x=1} - \int_0^1 \frac{2x}{1+x^2}\left(\int_0^x \frac{\ln^{q-1}(t)}{1-t^2}\mathrm{d}t\right)\mathrm{d}x$$

$$\overset{t=xu}{=} \ln(2)\int_0^1 \frac{\ln^{q-1}(t)}{1-t^2}\mathrm{d}t - 2\int_0^1\int_0^1 \frac{x^2\ln^{q-1}(xu)}{(1+x^2)(1-x^2u^2)}\mathrm{d}u\mathrm{d}x \tag{3.64}$$

$$\left\{\text{write } \frac{x^2}{(1+x^2)(1-x^2u^2)} = \frac{1}{(1+u^2)(1-x^2u^2)} - \frac{1}{(1+u^2)(1+x^2)}\right\}$$

$$= \ln(2)\int_0^1 \frac{\ln^{q-1}(t)}{1-t^2}\mathrm{d}t + 2\int_0^1\int_0^1 \frac{\ln^{q-1}(xu)}{(1+x^2)(1+u^2)}\mathrm{d}u\mathrm{d}x$$

$$\underbrace{-2\int_0^1\int_0^1 \frac{\ln^{q-1}(xu)}{(1+u^2)(1-x^2u^2)}\mathrm{d}u\mathrm{d}x}_{\text{interchange } x \text{ and } u}$$

$$= \ln(2)\int_0^1 \frac{\ln^{q-1}(t)}{1-t^2}\mathrm{d}t + 2\int_0^1\int_0^1 \frac{\ln^{q-1}(xu)}{(1+x^2)(1+u^2)}\mathrm{d}u\mathrm{d}x$$

$$-2\int_0^1\int_0^1 \frac{\ln^{q-1}(xu)}{(1+x^2)(1-x^2u^2)}\mathrm{d}u\mathrm{d}x$$

$$\{\text{add step (3.64) to both sides then divide by two}\}$$

$$= \ln(2)\int_0^1 \frac{\ln^{q-1}(t)}{1-t^2}\mathrm{d}t + \int_0^1\int_0^1 \frac{\ln^{q-1}(xu)}{(1+x^2)(1+u^2)}\mathrm{d}u\mathrm{d}x$$

$$-\int_0^1\int_0^1 \frac{\ln^{q-1}(xu)}{1-x^2u^2}\mathrm{d}u\mathrm{d}x.$$

First integral: Employ the result (3.14),

$$\int_0^1 \frac{\ln^{q-1}(t)}{1-t^2}\mathrm{d}t = (-1)^{q-1}(q-1)!\left(1-\frac{1}{2^q}\right)\zeta(q).$$

Second integral: Use the binomial theorem,

$$\begin{aligned}
\int_0^1\int_0^1 \frac{\ln^{q-1}(xu)}{(1+x^2)(1+u^2)}\mathrm{d}x\mathrm{d}u &= \int_0^1\int_0^1 \frac{[\ln(x)+\ln(u)]^{q-1}}{(1+x^2)(1+u^2)}\mathrm{d}x\mathrm{d}u \\
&= \int_0^1\int_0^1 \frac{\sum_{k=0}^{q-1}\binom{q-1}{k}\ln^{q-k-1}(u)\ln^k(x)}{(1+x^2)(1+u^2)}\mathrm{d}x\mathrm{d}u \\
&= \sum_{k=0}^{q-1}\binom{q-1}{k}\left(\int_0^1 \frac{\ln^{q-k-1}(u)}{1+u^2}\mathrm{d}u\right)\left(\int_0^1 \frac{\ln^k(x)}{1+x^2}\mathrm{d}x\right) \\
&\qquad \{\text{make use of the result (1.62)}\} \\
&= \sum_{k=0}^{q-1}\binom{q-1}{k}\left((-1)^{(q-k-1)}(q-k-1)!\,\beta(q-k)\right)\left((-1)^k k!\,\beta(k+1)\right) \\
&= (-1)^{q-1}(q-1)!\sum_{k=0}^{q-1}\beta(q-k)\beta(k+1).
\end{aligned}$$

Third integral: Let $x=\frac{t}{u}$,

$$\begin{aligned}
\int_0^1\int_0^1 \frac{\ln^{q-1}(xu)}{1-x^2u^2}\mathrm{d}x\mathrm{d}u &= \int_0^1\int_0^u \frac{\ln^{q-1}(t)}{u(1-t^2)}\mathrm{d}t\mathrm{d}u \\
&= \int_0^1 \frac{\ln^{q-1}(t)}{1-t^2}\left(\int_t^1 \frac{\mathrm{d}u}{u}\right)\mathrm{d}t \\
&= -\int_0^1 \frac{\ln^q(t)}{1-t^2}\mathrm{d}t \\
&= (-1)^{q-1}q!\left(1-\frac{1}{2^{q+1}}\right)\zeta(q+1).
\end{aligned}$$

Grouping the three integrals completes the proof.

Examples:

$$\int_0^1 \frac{\ln(x)\ln(1+x^2)}{1-x^2}\mathrm{d}x = \frac{7}{4}\zeta(3)-\frac{3}{4}\ln(2)\zeta(2)-\frac{\pi}{2}G;$$

$$\int_0^1 \frac{\ln^2(x)\ln(1+x^2)}{1-x^2}\mathrm{d}x = \frac{7}{4}\ln(2)\zeta(3)-\frac{45}{16}\zeta(4)+2G^2;$$

$$\int_0^1 \frac{\ln^3(x)\ln(1+x^2)}{1-x^2}\mathrm{d}x = \frac{93}{4}\zeta(5)-\frac{45}{8}\ln(2)\zeta(4)-12G\beta(3)-3\pi\beta(4);$$

$$\int_0^1 \frac{\ln^4(x)\ln(1+x^2)}{1-x^2}\mathrm{d}x = \frac{93}{4}\ln(2)\zeta(5) - \frac{12285}{128}\zeta(6) + 48G\beta(4) + 12\pi\beta(5).$$

3.1.34 $\int_0^1 x^{n-1}\operatorname{Li}_q(x)\mathrm{d}x$

Let $q \in \mathbb{Z}^+$. Then the following identity holds

$$\int_0^1 x^{n-1}\operatorname{Li}_q(x)\mathrm{d}x = (-1)^{q-1}\frac{H_n}{n^q} - \sum_{k=1}^{q-1}\frac{(-1)^k}{n^k}\zeta(q-k+1). \tag{3.65}$$

Proof. Let's evaluate the following integrals:

$$\begin{aligned}\int_0^1 x^{n-1}\operatorname{Li}_2(x)\mathrm{d}x &\overset{\text{IBP}}{=} \frac{x^n}{n}\operatorname{Li}_2(x)\Big|_0^1 + \frac{1}{n}\int_0^1 x^{n-1}\ln(1-x)\mathrm{d}x \\ &\{\text{recall the result (1.124)}\} \\ &= \frac{\zeta(2)}{n} + \frac{1}{n}\left(-\frac{H_n}{n}\right) \\ &= \frac{\zeta(2)}{n} - \frac{H_n}{n^2}. \end{aligned} \tag{3.66}$$

$$\begin{aligned}\int_0^1 x^{n-1}\operatorname{Li}_3(x)\mathrm{d}x &\overset{\text{IBP}}{=} \frac{x^n}{n}\operatorname{Li}_3(x)\Big|_0^1 - \frac{1}{n}\int_0^1 x^{n-1}\operatorname{Li}_2(x)\mathrm{d}x \\ &\{\text{substitute the result (3.66)}\} \\ &= \frac{\zeta(3)}{n} - \frac{1}{n}\left(\frac{\zeta(2)}{n} - \frac{H_n}{n^2}\right) \\ &= \frac{\zeta(3)}{n} - \frac{\zeta(2)}{n^2} + \frac{H_n}{n^3}. \end{aligned} \tag{3.67}$$

$$\begin{aligned}\int_0^1 x^{n-1}\operatorname{Li}_4(x)\mathrm{d}x &= \frac{x^n}{n}\operatorname{Li}_4(x)\Big|_0^1 - \frac{1}{n}\int_0^1 x^{n-1}\operatorname{Li}_3(x)\mathrm{d}x \\ &= \frac{\zeta(4)}{n} - \frac{1}{n}\left(\frac{\zeta(3)}{n} - \frac{\zeta(2)}{n^2} + \frac{H_n}{n^3}\right) \\ &= \frac{\zeta(4)}{n} - \frac{\zeta(3)}{n^2} + \frac{\zeta(2)}{n^3} - \frac{H_n}{n^4}. \end{aligned} \tag{3.68}$$

In general, we have

$$\int_0^1 x^{n-1}\operatorname{Li}_q(x)\mathrm{d}x=\frac{\zeta(q)}{n}-\frac{\zeta(q-1)}{n^2}+....+(-1)^q\frac{\zeta(2)}{n^{q-1}}+(-1)^{q-1}\frac{H_n}{n^q}$$
$$=\sum_{k=1}^{q-1}(-1)^{k-1}\frac{\zeta(q-k+1)}{n^k}+(-1)^{q-1}\frac{H_n}{n^q}.$$

3.1.35 $\int_0^1 \frac{\ln^{2q}(1-x)\operatorname{Li}_2(x)}{x}\mathrm{d}x$

Let $q\in\mathbb{Z}^+$. Then the following identity holds:

$$\int_0^1 \frac{\ln^{2q}(1-x)\operatorname{Li}_2(x)}{x}\mathrm{d}x=(2q)!\left(1-\frac{q(2q+1)}{2}\right)\zeta(2q+3)$$
$$+(2q)!\sum_{k=1}^{q}(2k-1)\zeta(2q-2k+3)\zeta(2k). \tag{3.69}$$

Proof. With subbing $1-x=y$, we have

$$\int_0^1 \frac{\ln^{2q}(1-x)\operatorname{Li}_2(x)}{x}\mathrm{d}x=\int_0^1 \frac{\ln^{2q}(y)\operatorname{Li}_2(1-y)}{1-y}\mathrm{d}y$$
$$\{\text{recall the dilogarithm reflection formula (1.80)}\}$$
$$=\int_0^1 \frac{\ln^{2q}(y)}{1-y}(\zeta(2)-\ln(y)\ln(1-y)-\operatorname{Li}_2(y))\mathrm{d}y$$
$$=\zeta(2)\int_0^1\frac{\ln^{2q}(y)}{1-y}\mathrm{d}y-\int_0^1\frac{\ln^{2q+1}(y)\ln(1-y)}{1-y}\mathrm{d}y-\int_0^1\frac{\ln^{2q}(y)\operatorname{Li}_2(y)}{1-y}\mathrm{d}y.$$

The first integral is given in (3.8). For the second, let $q\to q+1$ in (3.43),

$$\int_0^1\frac{\ln^{2q+1}(x)\ln(1-x)}{1-x}\mathrm{d}x=\frac{1}{2}(2q+2)!\,\zeta(2q+3)$$
$$-(2q+1)!\sum_{k=1}^{q}\zeta(2q-2k+3)\zeta(2k).$$

For the third integral, expand $\frac{\operatorname{Li}_2(y)}{1-y}$ in series as shown in (2.3),

$$\int_0^1\frac{\ln^{2q}(y)\operatorname{Li}_2(y)}{1-y}\mathrm{d}y=\sum_{n=1}^{\infty}H_{n-1}^{(2)}\int_0^1 y^{n-1}\ln^{2q}(y)\mathrm{d}y$$

$$= \sum_{n=1}^{\infty} \left(H_n^{(2)} - \frac{1}{n^2} \right) \left(\frac{(2q)!}{n^{2q+1}} \right)$$

$$= (2q)! \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^{2q+1}} - (2q)!\, \zeta(2q+3)$$

$$\{\text{replace } q \text{ with } q+1 \text{ in (4.21) to get this sum}\}$$

$$= (2q)!\, \zeta(2)\zeta(2q+1) - \left(\frac{1}{2}(q+2)(2q+1) + 1 \right) (2q)!\, \zeta(2q+3)$$

$$+2(2q)! \sum_{k=1}^{q} k\, \zeta(2q-2k+2)\zeta(2k+1)$$

$$\{\text{reverse the order of the sum terms using (1.2)}\}$$

$$= (2q)!\, \zeta(2)\zeta(2q+1) - \left(\frac{1}{2}(q+2)(2q+1) + 1 \right) (2q)!\, \zeta(2q+3)$$

$$+2(2q)! \sum_{k=1}^{q} (q-k+1)\, \zeta(2q-2k+3)\zeta(2k).$$

Group the three integrals to finalize the proof.

Examples:

$$\int_0^1 \frac{\ln^2(1-x)\operatorname{Li}_2(x)}{x} \mathrm{d}x = 2\zeta(2)\zeta(3) - \zeta(5);$$

$$\int_0^1 \frac{\ln^4(1-x)\operatorname{Li}_2(x)}{x} \mathrm{d}x = 24\zeta(2)\zeta(5) + 72\zeta(3)\zeta(4) - 96\zeta(7);$$

$$\int_0^1 \frac{\ln^6(1-x)\operatorname{Li}_2(x)}{x} \mathrm{d}x = 360\zeta(3)\zeta(6) + 2160\zeta(4)\zeta(5) + 7200\zeta(2)\zeta(7)$$
$$-6840\zeta(9);$$

$$\int_0^1 \frac{\ln^8(1-x)\operatorname{Li}_2(x)}{x} \mathrm{d}x = 282240\zeta(3)\zeta(8) + 120960\zeta(4)\zeta(7) + 201600\zeta(5)\zeta(6)$$
$$+40320\zeta(2)\zeta(9) - 685440\zeta(11).$$

3.1.36 $\int_0^\infty \frac{\operatorname{Li}_q(-x^2)}{1+x^2} \mathrm{d}x$

Let $q \in \mathbb{Z}^+$. Then the following identity holds:

$$\int_0^\infty \frac{\operatorname{Li}_q(-x^2)}{1+x^2} \mathrm{d}x = -2^{q-1}\pi\, \eta(q). \tag{3.70}$$

Proof. By writing the integral representation of $\mathrm{Li}_q(-x^2)$, we have

$$\begin{aligned}\int_0^\infty \frac{\mathrm{Li}_q(-x^2)}{1+x^2}\mathrm{d}x &= \int_0^\infty \frac{1}{1+x^2}\left(\frac{(-1)^q}{(q-1)!}\int_0^1 \frac{x^2\ln^{q-1}(y)}{1+x^2y}\mathrm{d}y\right)\mathrm{d}x\\ &= \frac{(-1)^q}{(q-1)!}\int_0^1 \ln^{q-1}(y)\left(\int_0^\infty \frac{x^2}{(1+x^2)(1+x^2y)}\mathrm{d}x\right)\mathrm{d}y\\ &= \frac{(-1)^q}{(q-1)!}\int_0^1 \ln^{q-1}(y)\left(\frac{\pi}{2}\frac{1}{y+\sqrt{y}}\right)\mathrm{d}y\\ &\overset{\sqrt{y}=x}{=} \frac{(-1)^q 2^{q-1}\pi}{(q-1)!}\int_0^1 \frac{\ln^{q-1}(x)}{1+x}\mathrm{d}x.\end{aligned}$$

This integral is given in (1.53).

Examples:

$$\begin{aligned}\int_0^\infty \frac{\mathrm{Li}_1(-x^2)}{1+x^2}\mathrm{d}x &= -\pi\ln(2);\\ \int_0^\infty \frac{\mathrm{Li}_2(-x^2)}{1+x^2}\mathrm{d}x &= -\frac{\pi^3}{6};\\ \int_0^\infty \frac{\mathrm{Li}_3(-x^2)}{1+x^2}\mathrm{d}x &= -3\pi\,\zeta(3);\\ \int_0^\infty \frac{\mathrm{Li}_4(-x^2)}{1+x^2}\mathrm{d}x &= -\frac{7\pi^5}{90}.\end{aligned}$$

3.1.37 $\int_0^\infty \frac{\mathrm{Li}_q(-x)}{1+x^2}\mathrm{d}x$

Let $q \in \mathbb{Z}^+$. Then the following identity holds:

$$\int_0^\infty \frac{\mathrm{Li}_q(-x)}{1+x^2}\mathrm{d}x = \frac{-\pi}{2^{q+1}}\eta(q) - q\,\beta(q+1). \tag{3.71}$$

Proof. Following the previous proof,

$$\begin{aligned}\int_0^\infty \frac{\mathrm{Li}_q(-x)}{1+x^2}\mathrm{d}x &= \int_0^\infty \frac{1}{1+x^2}\left(\frac{(-1)^q}{(q-1)!}\int_0^1 \frac{x\ln^{q-1}(y)}{1+xy}\mathrm{d}y\right)\mathrm{d}x\\ &= \frac{(-1)^q}{(q-1)!}\int_0^1 \ln^{q-1}(y)\left(\int_0^\infty \frac{x}{(1+x^2)(1+xy)}\mathrm{d}x\right)\mathrm{d}y\\ &= \frac{(-1)^q}{(q-1)!}\int_0^1 \ln^{q-1}(y)\left(\frac{\pi}{2}\frac{y}{1+y^2} - \frac{\ln(y)}{1+y^2}\right)\mathrm{d}y\end{aligned}$$

$$= \frac{(-1)^q \pi}{2(q-1)!} \int_0^1 \frac{y \ln^{q-1}(y)}{1+y^2} \mathrm{d}y - \frac{(-1)^q}{(q-1)!} \int_0^1 \frac{\ln^q(y)}{1+y^2} \mathrm{d}y.$$

These two integrals are given in (3.13) and (1.62) respectively.

Examples:

$$\int_0^\infty \frac{\mathrm{Li}_1(-x)}{1+x^2} \mathrm{d}x = -\frac{\pi}{4} \ln(2) - G;$$
$$\int_0^\infty \frac{\mathrm{Li}_2(-x)}{1+x^2} \mathrm{d}x = -\frac{\pi^3}{96} - 2\beta(3);$$
$$\int_0^\infty \frac{\mathrm{Li}_3(-x)}{1+x^2} \mathrm{d}x = -\frac{3\pi}{64} \zeta(3) - 3\beta(4);$$
$$\int_0^\infty \frac{\mathrm{Li}_4(-x)}{1+x^2} \mathrm{d}x = -\frac{7\pi^5}{23040} - 4\beta(5).$$

3.1.38 $\int_0^1 \frac{\mathrm{Li}_q(-x)}{1+x} \mathrm{d}x$

Let $q \in \mathbb{Z}_{>1}$. Then the following identity holds:

$$\int_0^1 \frac{\mathrm{Li}_q(-x)}{1+x} \mathrm{d}x = \left(\frac{q}{2} + \frac{1}{2^q} - 1\right) \zeta(q+1) - \ln(2)\eta(q) - \frac{1}{2} \sum_{k=1}^{q-2} \eta(k+1)\eta(q-k). \tag{3.72}$$

Proof. Using $\frac{1}{1+x} = \frac{1}{x} - \frac{1}{x(1+x)}$, we have

$$\int_0^1 \frac{\mathrm{Li}_q(-x)}{1+x} \mathrm{d}x = \int_0^1 \frac{\mathrm{Li}_q(-x)}{x} \mathrm{d}x - \int_0^1 \frac{\mathrm{Li}_q(-x)}{x(1+x)} \mathrm{d}x$$
$$= \mathrm{Li}_{q+1}(-x)\big|_0^1 - \int_0^1 \frac{1}{1+x} \left(\frac{(-1)^q}{(q-1)!} \int_0^1 \frac{\ln^{q-1}(y)}{1+xy} \mathrm{d}y \right) \mathrm{d}x$$
$$= \mathrm{Li}_{q+1}(-1) - \frac{(-1)^q}{(q-1)!} \int_0^1 \ln^{q-1}(y) \left(\int_0^1 \frac{\mathrm{d}x}{(1+x)(1+yx)} \right) \mathrm{d}y$$
$$= -(1-2^{-q})\zeta(q+1) - \frac{(-1)^q}{(q-1)!} \int_0^1 \frac{\ln^{q-1}(y) \ln\left(\frac{2}{1+y}\right)}{1-y} \mathrm{d}y.$$

This integral is calculated in (3.47).

Examples:

$$\int_0^1 \frac{\mathrm{Li}_2(-x)}{1+x}\mathrm{d}x = \frac{1}{4}\zeta(3) - \frac{1}{2}\ln(2)\zeta(2);$$
$$\int_0^1 \frac{\mathrm{Li}_3(-x)}{1+x}\mathrm{d}x = \frac{5}{16}\zeta(4) - \frac{3}{4}\ln(2)\zeta(3);$$
$$\int_0^1 \frac{\mathrm{Li}_4(-x)}{1+x}\mathrm{d}x = \frac{17}{16}\zeta(5) - \frac{7}{8}\ln(2)\zeta(4) - \frac{3}{8}\zeta(2)\zeta(3);$$
$$\int_0^1 \frac{\mathrm{Li}_5(-x)}{1+x}\mathrm{d}x = \frac{49}{64}\zeta(6) - \frac{9}{32}\zeta^2(3) - \frac{15}{16}\ln(2)\zeta(5).$$

3.1.39 $\int_0^1 \frac{\mathrm{Li}_{2q}(x)}{1+x}\mathrm{d}x$

Let $q \in \mathbb{Z}^+$. Then the following identity holds:

$$\int_0^1 \frac{\mathrm{Li}_{2q}(x)}{1+x}\mathrm{d}x = \ln(2)\zeta(2q) - \left(q - \frac{2q+1}{2^{2q+1}}\right)\zeta(2q+1) + \sum_{k=1}^{q-1}(1-2^{2k-2q})\zeta(2k)\zeta(2q-2k+1). \tag{3.73}$$

Proof.

$$\begin{aligned}
\int_0^1 \frac{\mathrm{Li}_{2q}(x)}{1+x}\mathrm{d}x &= \int_0^1 \frac{\mathrm{Li}_{2q}(x)}{x}\mathrm{d}x - \int_0^1 \frac{\mathrm{Li}_{2q}(x)}{x(1+x)}\mathrm{d}x \\
&= \mathrm{Li}_{2q+1}(x)\big|_0^1 - \int_0^1 \frac{1}{1+x}\left(\frac{-1}{(2q-1)!}\int_0^1 \frac{\ln^{2q-1}(y)}{1-xy}\mathrm{d}y\right)\mathrm{d}x \\
&= \mathrm{Li}_{2q+1}(1) + \frac{1}{(2q-1)!}\int_0^1 \ln^{2q-1}(y)\left(\int_0^1 \frac{\mathrm{d}x}{(1+x)(1-yx)}\right)\mathrm{d}y \\
&= \zeta(2q+1) + \frac{1}{(2q-1)!}\int_0^1 \ln^{2q-1}(y)\left(\frac{\ln(2)-\ln(1-y)}{1+y}\right)\mathrm{d}y.
\end{aligned}$$

These two integrals are given in (3.11) and (3.57) respectively.

Examples:

$$\int_0^1 \frac{\mathrm{Li}_2(x)}{1+x}\mathrm{d}x = \ln(2)\zeta(2) - \frac{5}{8}\zeta(3);$$
$$\int_0^1 \frac{\mathrm{Li}_4(x)}{1+x}\mathrm{d}x = \ln(2)\zeta(4) + \frac{3}{4}\zeta(2)\zeta(3) - \frac{59}{32}\zeta(5);$$

$$\int_0^1 \frac{\operatorname{Li}_6(x)}{1+x}\mathrm{d}x = \ln(2)\zeta(6) + \frac{3}{4}\zeta(3)\zeta(4) + \frac{15}{16}\zeta(2)\zeta(5) - \frac{377}{128}\zeta(7);$$

$$\int_0^1 \frac{\operatorname{Li}_8(x)}{1+x}\mathrm{d}x = \ln(2)\zeta(8) + \frac{3}{4}\zeta(3)\zeta(6) + \frac{15}{16}\zeta(4)\zeta(5) + \frac{63}{64}\zeta(2)\zeta(7) - \frac{2039}{512}\zeta(9).$$

3.1.40 $\int_0^1 \frac{\operatorname{Li}_{2q+1}(-x)}{1+x^2}\mathrm{d}x$

Let $q \in \mathbb{Z}_{\geq 0}$. Then the following identity holds:

$$\int_0^1 \frac{\operatorname{Li}_{2q+1}(-x)}{1+x^2}\mathrm{d}x = \frac{-\pi}{2^{2q+3}}\eta(2q+1) - \frac{1}{2}(2q+1)\beta(2q+2) + \sum_{k=0}^{q} \eta(2q-2k)\beta(2k+2). \tag{3.74}$$

Proof.

$$\begin{aligned}
&(2p)! \int_0^1 \frac{\operatorname{Li}_{2q+1}(-x)}{1+x^2}\mathrm{d}x \\
&= \int_0^1 \frac{1}{1+x^2}\left(\int_0^1 \frac{-x\ln^{2q}(y)}{1+xy}\mathrm{d}y\right)\mathrm{d}x \\
&= \int_0^1 \ln^{2q}(y)\left(\int_0^1 \frac{-x}{(1+x^2)(1+xy)}\mathrm{d}x\right)\mathrm{d}y \\
&= \int_0^1 \ln^{2q}(y)\left(\frac{\ln(1+y)}{1+y^2} - \frac{\ln(2)}{2}\frac{1}{1+y^2} - \frac{\pi}{4}\frac{y}{1+y^2}\right)\mathrm{d}y \\
&= \int_0^1 \frac{\ln^{2q}(y)\ln(1+y)}{1+y^2}\mathrm{d}y - \frac{\ln(2)}{2}\int_0^1 \frac{\ln^{2q}(y)}{1+y^2}\mathrm{d}y - \frac{\pi}{4}\int_0^1 \frac{y\ln^{2q}(y)}{1+y^2}\mathrm{d}y.
\end{aligned}$$

Collect the results (3.55), (1.62), and (3.13) to finish the proof.

Examples:

$$\int_0^1 \frac{\operatorname{Li}_1(-x)}{1+x^2}\mathrm{d}x = -\frac{\pi}{8}\ln(2);$$

$$\int_0^1 \frac{\operatorname{Li}_3(-x)}{1+x^2}\mathrm{d}x = \frac{1}{2}G\zeta(2) - \frac{3\pi}{128}\zeta(3) - \beta(4);$$

$$\int_0^1 \frac{\operatorname{Li}_5(-x)}{1+x^2}\mathrm{d}x = \frac{7}{8}G\zeta(4) + \frac{1}{2}\zeta(2)\beta(4) - \frac{15\pi}{2048}\zeta(5) - 2\beta(6);$$

$$\int_0^1 \frac{\operatorname{Li}_7(-x)}{1+x^2}\mathrm{d}x = \frac{31}{32}G\zeta(6) + \frac{7}{8}\zeta(4)\beta(4) + \frac{1}{2}\zeta(2)\beta(6) - \frac{63\pi}{32768}\zeta(7) - 3\beta(8).$$

Note that $\int_0^1 \frac{\mathrm{Li}_1(-x)}{1+x^2}\mathrm{d}x = -\int_0^1 \frac{\ln(1+x)}{1+x^2}\mathrm{d}x$, which is also calculated in (3.96).

3.1.41 $\int_0^1 \frac{\mathrm{Li}_{2q+1}(x)}{1+x^2}\mathrm{d}x$

Let $q \in \mathbb{Z}_{\geq 0}$. Then the following identity holds:

$$\int_0^1 \frac{\mathrm{Li}_{2q+1}(x)}{1+x^2}\mathrm{d}x = \frac{1}{2}(2q+1)\beta(2q+2) - \frac{\pi}{2^{2q+3}}\eta(2q+1) - \sum_{k=0}^{q}\zeta(2q-2k)\beta(2k+2). \tag{3.75}$$

Proof.

$$\begin{aligned}
&(2q)!\int_0^1 \frac{\mathrm{Li}_{2q+1}(x)}{1+x^2}\mathrm{d}x \\
&= \int_0^1 \frac{1}{1+x^2}\left(\int_0^1 \frac{x\ln^{2q}(y)}{1-xy}\mathrm{d}y\right)\mathrm{d}x \\
&= \int_0^1 \ln^{2q}(y)\left(\int_0^1 \frac{x}{(1+x^2)(1-xy)}\mathrm{d}x\right)\mathrm{d}y \\
&= \int_0^1 \ln^{2q}(y)\left(\frac{\ln(2)}{2(1+y^2)} - \frac{\pi\, y}{4(1+y^2)} - \frac{\ln(1-y)}{1+y^2}\right)\mathrm{d}y \\
&= \frac{\ln(2)}{2}\int_0^1 \frac{\ln^{2q}(y)}{1+y^2}\mathrm{d}y - \frac{\pi}{4}\int_0^1 \frac{y\ln^{2q}(y)}{1+y^2}\mathrm{d}y - \int_0^1 \frac{\ln^{2q}(y)\ln(1-y)}{1+y^2}\mathrm{d}y.
\end{aligned}$$

These three integral are given in (1.62), (3.13), and (3.57) respectively.

Examples:

$$\int_0^1 \frac{\mathrm{Li}_1(x)}{1+x^2}\mathrm{d}x = G - \frac{\pi}{8}\ln(2);$$

$$\int_0^1 \frac{\mathrm{Li}_3(x)}{1+x^2}\mathrm{d}x = 2\beta(4) - G\zeta(2) - \frac{3\pi}{128}\zeta(3);$$

$$\int_0^1 \frac{\mathrm{Li}_5(x)}{1+x^2}\mathrm{d}x = 3\beta(6) - \beta(4)\zeta(2) - G\zeta(4) - \frac{15\pi}{2048}\zeta(5);$$

$$\int_0^1 \frac{\mathrm{Li}_7(x)}{1+x^2}\mathrm{d}x = 4\beta(8) - \beta(6)\zeta(2) - \beta(4)\zeta(4) - G\zeta(6) - \frac{63\pi}{32768}\zeta(7).$$

3.1.42 $\int_0^1 \frac{\mathrm{Li}_{2q+1}(-x^2)}{1+x^2}\mathrm{d}x$

Let $q \in \mathbb{Z}_{\geq 0}$. Then the following identity holds:

$$\int_0^1 \frac{\mathrm{Li}_{2q+1}(-x^2)}{1+x^2}\mathrm{d}x = -2^{2q-1}\pi\,\eta(2q+1) + 2\sum_{k=0}^{q} 2^{2k}\eta(2q-2k)\beta(2k+2). \tag{3.76}$$

Proof.

$$\begin{aligned}
&(2q)!\int_0^1 \frac{\mathrm{Li}_{2q+1}(-x^2)}{1+x^2}\mathrm{d}x\\
&= \int_0^1 \frac{1}{1+x^2}\left(\int_0^1 \frac{-x^2\ln^{2q}(y)}{(1+x^2)(1+x^2y)}\mathrm{d}y\right)\mathrm{d}x\\
&= \int_0^1 \ln^{2q}(y)\left(\int_0^1 \frac{-x^2}{(1+x^2)(1+x^2y)}\mathrm{d}x\right)\mathrm{d}y\\
&= \int_0^1 \ln^{2q}(y)\left(\frac{\pi}{4}\frac{1}{1-y} - \frac{\arctan\sqrt{y}}{\sqrt{y}(1-y)}\right)\mathrm{d}y\\
&= \frac{\pi}{4}\int_0^1 \frac{\ln^{2q}(y)}{1-y}\mathrm{d}y - \int_0^1 \frac{\ln^{2q}(y)\arctan(\sqrt{y})}{\sqrt{y}(1-y)}\mathrm{d}y\\
&\overset{\sqrt{y}=x}{=} \frac{\pi}{4}\int_0^1 \frac{\ln^{2q}(y)}{1-y}\mathrm{d}y - 2^{2q+1}\int_0^1 \frac{\ln^{2q}(x)\arctan(x)}{1-x^2}\mathrm{d}x.
\end{aligned}$$

These two integrals are given in (3.8) and (3.60).

Examples:

$$\int_0^1 \frac{\mathrm{Li}_1(-x^2)}{1+x^2}\mathrm{d}x = G - \frac{\pi}{2}\ln(2);$$

$$\int_0^1 \frac{\mathrm{Li}_3(-x^2)}{1+x^2}\mathrm{d}x = 4\beta(4) + G\zeta(2) - \frac{3}{2}\pi\zeta(3);$$

$$\int_0^1 \frac{\mathrm{Li}_5(-x^2)}{1+x^2}\mathrm{d}x = 16\beta(6) + 4\beta(4)\zeta(2) + \frac{7}{4}G\zeta(4) - \frac{15}{2}\pi\zeta(5);$$

$$\int_0^1 \frac{\mathrm{Li}_7(-x^2)}{1+x^2}\mathrm{d}x = 64\beta(8) + 16\beta(6)\zeta(2) + 7\beta(4)\zeta(4) + \frac{31}{16}G\zeta(6) - \frac{63}{2}\pi\zeta(7).$$

3.1.43 $\int_0^\infty \frac{\ln^{2q}(x)}{1+yx^2}\mathrm{d}x$

Let $q \in \mathbb{Z}_{\geq 0}$. Then the following identity holds:

$$\int_0^\infty \frac{\ln^{2q}(x)}{1+yx^2}\mathrm{d}x = \frac{\left(\frac{\pi}{2}\right)^{2q+1}}{\sqrt{y}} \sum_{j=0}^{q} \binom{2q}{2j} \frac{|E_{2q-2j}|}{\pi^{2j}} \ln^{2j}(y). \tag{3.77}$$

Proof. Make the change of variable $x = \frac{t}{\sqrt{y}}$, we have

$$\int_0^\infty \frac{\ln^{2q}(x)}{1+yx^2}\mathrm{d}x = \frac{1}{\sqrt{y}}\int_0^\infty (\ln(\sqrt{y}) - \ln(t))^{2q}\,\frac{\mathrm{d}t}{1+t^2}$$

$$\left\{\text{use the binomial theorem } (x+y)^a = \sum_{j=0}^{a}\binom{a}{j}x^j y^{a-j}\right\}$$

$$= \frac{1}{\sqrt{y}}\int_0^\infty \left(\sum_{j=0}^{2q}\binom{2q}{j}\ln^j(\sqrt{y})(-\ln(t))^{2q-j}\right)\frac{\mathrm{d}t}{1+t^2}$$

$$= \frac{1}{\sqrt{y}}\sum_{j=0}^{2q}\binom{2q}{j}\ln^j(\sqrt{y})\int_0^\infty \frac{(-\ln(t))^{2q-j}}{1+t^2}\mathrm{d}t$$

$$\left\{\text{use} \sum_{j=0}^{2q} f(j) = \sum_{j=0}^{q} f(2j) + \sum_{j=0}^{q} f(2j+1)\right\}$$

$$= \frac{1}{\sqrt{y}}\sum_{j=0}^{q}\binom{2q}{2j}\ln^{2j}(\sqrt{y})\int_0^\infty \frac{\ln^{2q-2j}(t)}{1+t^2}\mathrm{d}t$$

$$+\frac{1}{\sqrt{y}}\sum_{j=0}^{q}\binom{2q}{2j+1}\ln^{2j+1}(\sqrt{y})\int_0^\infty \frac{\ln^{2q-2j-1}(t)}{1+t^2}\mathrm{d}t$$

$\{$based on (3.22), ignore the second sum since the power of $\ln(t)$ is odd$\}$

$$= \frac{1}{\sqrt{y}}\sum_{j=0}^{q}\binom{2q}{2j}\ln^{2j}(\sqrt{y})\int_0^\infty \frac{\ln^{2q-2j}(t)}{1+t^2}\mathrm{d}t.$$

This integral is found in (3.21).

3.1.44 $\int_0^\infty \frac{\ln^{2q-1}(x)}{1+yx^2}\mathrm{d}x$

Let $q \in \mathbb{Z}^+$. Then the following identity holds:

$$\int_0^\infty \frac{\ln^{2q-1}(x)}{1+yx^2}\mathrm{d}x = \frac{-\left(\frac{\pi}{2}\right)^{2q-1}}{2\sqrt{y}} \sum_{j=0}^{q-1} \binom{2q-1}{2j+1} \frac{|E_{2q-2j-2}|}{\pi^{2j}} \ln^{2j+1}(y). \tag{3.78}$$

Proof. Applying the previous proof, we have

$$\int_0^\infty \frac{\ln^{2q-1}(x)}{1+yx^2}\mathrm{d}x = \frac{-1}{\sqrt{y}} \int_0^\infty \left(\ln(\sqrt{y}) - \ln(t)\right)^{2q-1} \frac{\mathrm{d}t}{1+t^2}$$

$$= \frac{-1}{\sqrt{y}} \int_0^\infty \left(\sum_{j=0}^{2q-1} \binom{2q-1}{j} \ln^j(\sqrt{y})(-\ln(t))^{2q-j-1} \right) \frac{\mathrm{d}t}{1+t^2}$$

$$= \frac{-1}{\sqrt{y}} \sum_{j=0}^{2q-1} \binom{2q-1}{j} \ln^j(\sqrt{y}) \int_0^\infty \frac{(-\ln(t))^{2q-j-1}}{1+t^2}\mathrm{d}t$$

$$\left\{ \text{use } \sum_{j=0}^{2q-1} f(j) = \sum_{j=0}^{q-1} f(2j) + \sum_{j=0}^{q-1} f(2j+1) \right\}$$

$$= \frac{-1}{\sqrt{y}} \sum_{j=0}^{q-1} \binom{2q-1}{2j} \ln^{2j}(\sqrt{y}) \int_0^\infty \frac{\ln^{2q-2j-1}(t)}{1+t^2}\mathrm{d}t$$

$$- \frac{1}{\sqrt{y}} \sum_{j=0}^{q-1} \binom{2q-1}{2j+1} \ln^{2j+1}(\sqrt{y}) \int_0^\infty \frac{\ln^{2q-2j-2}(t)}{1+t^2}\mathrm{d}t.$$

Ignore the first sum and the proof follows on substituting (3.21).

3.1.45 $\int_0^\infty \frac{\ln^{2q}(x)\,\mathrm{Li}_{2p+1}(-x^2)}{1+x^2}\mathrm{d}x$

Let $q, p \in \mathbb{Z}^+$. Then the following identity holds:

$$\int_0^\infty \frac{\ln^{2q}(x)\,\mathrm{Li}_{2p+1}(-x^2)}{1+x^2}\mathrm{d}x = |E_{2q}| \left(\frac{\pi}{2}\right)^{2q+1} \zeta(2p+1)$$

(continued)

$$-\frac{\left(\frac{\pi}{2}\right)^{2q+1}}{(2p)!}\sum_{j=0}^{q}\binom{2q}{2j}\frac{(2p+2j)!}{\pi^{2j}}|E_{2q-2j}|(2^{2j+2p+1}-1)\zeta(2j+2p+1). \tag{3.79}$$

Proof.

$$\begin{aligned}
&\int_0^\infty \frac{\ln^{2q}(x)\operatorname{Li}_{2p+1}(-x^2)}{1+x^2}\mathrm{d}x\\
&= \int_0^\infty \frac{\ln^{2q}(x)}{1+x^2}\left(\frac{1}{(2p)!}\int_0^1 \frac{-x^2\ln^{2p}(y)}{1+yx^2}\mathrm{d}y\right)\mathrm{d}x\\
&= \frac{1}{(2p)!}\int_0^1 \ln^{2p}(y)\left(\int_0^\infty \frac{-x^2\ln^{2q}(x)}{(1+x^2)(1+yx^2)}\mathrm{d}x\right)\mathrm{d}y\\
&= \frac{1}{(2p)!}\int_0^1 \frac{\ln^{2p}(y)}{1-y}\left(\int_0^\infty \frac{\ln^{2q}(x)}{1+x^2}\mathrm{d}x-\int_0^\infty \frac{\ln^{2q}(x)}{1+yx^2}\mathrm{d}x\right)\mathrm{d}y\\
&\{\text{these two integrals are given in (3.21) and (3.77)}\}\\
&= \frac{|E_{2q}|}{(2p)!}\left(\frac{\pi}{2}\right)^{2q+1}\int_0^1 \frac{\ln^{2p}(y)}{1-y}\mathrm{d}y\\
&-\frac{\left(\frac{\pi}{2}\right)^{2q+1}}{(2p)!}\sum_{j=0}^{q}\binom{2q}{2j}|E_{2q-2j}|\pi^{-2j}\int_0^1 \frac{\ln^{2j+2p}(y)}{\sqrt{y}(1-y)}\mathrm{d}y\\
&\{\text{let } \sqrt{y}=x \text{ in the second integral}\}\\
&= \frac{|E_{2q}|}{(2p)!}\left(\frac{\pi}{2}\right)^{2q+1}\int_0^1 \frac{\ln^{2p}(y)}{1-y}\mathrm{d}y\\
&-\frac{\left(\frac{\pi}{2}\right)^{2q+1}}{(2p)!}\sum_{j=0}^{q}\binom{2q}{2j}|E_{2q-2j}|\pi^{-2j}2^{2j+2p+1}\int_0^1 \frac{\ln^{2j+2p}(x)}{1-x^2}\mathrm{d}x.
\end{aligned}$$

The proof completes on employing (3.8) and (3.14).

Examples:

$$\int_0^\infty \frac{\ln^2(x)\operatorname{Li}_3(-x^2)}{1+x^2}\mathrm{d}x = -\frac{93}{2}\pi\zeta(5)-\frac{3}{4}\pi^3\zeta(3);$$

$$\int_0^\infty \frac{\ln^4(x)\operatorname{Li}_3(-x^2)}{1+x^2}\mathrm{d}x = -\frac{5715}{4}\pi\zeta(7)-\frac{279}{4}\pi^3\zeta(5)-\frac{15}{16}\pi^5\zeta(3);$$

$$\int_0^\infty \frac{\ln^4(x)\operatorname{Li}_5(-x^2)}{1+x^2}\mathrm{d}x = -\frac{53655}{2}\pi\zeta(9)-\frac{5715}{8}\pi^3\zeta(7)-\frac{75}{16}\pi^5\zeta(5);$$

$$\int_0^\infty \frac{\ln^6(x)\operatorname{Li}_5(-x^2)}{1+x^2}\mathrm{d}x = -\frac{9672075}{4}\pi\zeta(11)-\frac{804825}{8}\pi^3\zeta(9)-\frac{142875}{64}\pi^5\zeta(7)$$

$$-\frac{915}{64}\pi^7\zeta(5).$$

3.1.46 $\int_0^\infty \frac{\ln^{2q-1}(x)\operatorname{Li}_{2p}(-x^2)}{1+x^2}\mathrm{d}x$

Let $q, p \in \mathbb{Z}^+$. Then the following identity holds:

$$\int_0^\infty \frac{\ln^{2q-1}(x)\operatorname{Li}_{2p}(-x^2)}{1+x^2}\mathrm{d}x = \frac{-\left(\frac{\pi}{2}\right)^{2q-1}}{2(2p-1)!}*$$

$$\sum_{j=0}^{q-1}\binom{2q-1}{2j+1}\frac{(2p+2j)!}{\pi^{2j}}|E_{2q-2j-2}|(2^{2j+2p+1}-1)\zeta(2j+2p+1). \quad (3.80)$$

Proof. Applying the previous proof, we have

$$\begin{aligned}
&\int_0^\infty \frac{\ln^{2q-1}(x)\operatorname{Li}_{2p}(-x^2)}{1+x^2}\mathrm{d}x \\
&= \int_0^\infty \frac{\ln^{2q-1}(x)}{1+x^2}\left(\int_0^1 \frac{x^2\ln^{2p-1}(y)}{(2p-1)!(1+yx^2)}\mathrm{d}y\right)\mathrm{d}x \\
&= \frac{1}{(2p-1)!}\int_0^1 \ln^{2p-1}(y)\left(\int_0^\infty \frac{x^2\ln^{2q-1}(x)}{(1+x^2)(1+yx^2)}\mathrm{d}x\right)\mathrm{d}y \\
&= \frac{1}{(2p-1)!}\int_0^1 \frac{\ln^{2p-1}(y)}{1-y}\left(\int_0^\infty \frac{\ln^{2q-1}(x)}{1+yx^2}\mathrm{d}x - \int_0^\infty \frac{\ln^{2q-1}(x)}{1+x^2}\mathrm{d}x\right)\mathrm{d}y \\
&\qquad \{\text{the first integral is given in (3.78) and the second one is zero}\} \\
&= \frac{-\left(\frac{\pi}{2}\right)^{2q-1}}{2(2p-1)!}\sum_{j=0}^{q-1}\binom{2q-1}{2j+1}|E_{2q-2j-2}|\pi^{-2j}\int_0^1 \frac{\ln^{2j+2p}(y)}{\sqrt{y}(1-y)}\mathrm{d}y \\
&\overset{\sqrt{y}=x}{=} \frac{-\left(\frac{\pi}{2}\right)^{2q-1}}{(2p-1)!}\sum_{j=0}^{q-1}\binom{2q-1}{2j+1}|E_{2q-2j-2}|\pi^{-2j}2^{2j+2p}\int_0^1 \frac{\ln^{2j+2p}(x)}{1-x^2}\mathrm{d}x
\end{aligned}$$

and the proof completes on using (3.14).

Examples:

$$\int_0^\infty \frac{\ln(x)\operatorname{Li}_2(-x^2)}{1+x^2}\mathrm{d}x = -\frac{7}{2}\pi\zeta(3);$$

$$\int_0^\infty \frac{\ln^3(x)\operatorname{Li}_4(-x^2)}{1+x^2}\mathrm{d}x = -\frac{1905}{2}\pi\zeta(7) - \frac{93}{4}\pi^3\zeta(5);$$

$$\int_0^\infty \frac{\ln^5(x)\operatorname{Li}_4(-x^2)}{1+x^2}\mathrm{d}x = -53655\pi\zeta(9) - \frac{9525}{4}\pi^3\zeta(7) - \frac{775}{16}\pi^5\zeta(5);$$

$$\int_0^\infty \frac{\ln^5(x)\operatorname{Li}_6(-x^2)}{1+x^2}\mathrm{d}x = -\frac{1934415}{2}\pi\zeta(11) - \frac{53655}{2}\pi^3\zeta(9) - \frac{9525}{32}\pi^5\zeta(7).$$

3.1.47 $\int_0^1 \frac{\ln^{2q}(x)\operatorname{Li}_{2p+1}(-x^2)}{1+x^2}\mathrm{d}x$

Let $q, p \in \mathbb{Z}^+$. then the following identity holds:

$$\int_0^1 \frac{\ln^{2q}(x)\operatorname{Li}_{2p+1}(-x^2)}{1+x^2}\mathrm{d}x = \frac{1}{2}|E_{2q}|\left(\frac{\pi}{2}\right)^{2q+1}\zeta(2p+1)$$
$$-\frac{\left(\frac{\pi}{2}\right)^{2q+1}}{2(2p)!}\sum_{j=0}^{q}\binom{2q}{2j}\frac{(2p+2j)!}{\pi^{2j}}|E_{2q-2j}|(2^{2j+2p+1}-1)\zeta(2j+2p+1)$$
$$+2\sum_{k=0}^{p}\frac{(2k+2q+1)!}{(2k+1)!}2^{2k}\eta(2p-2k)\beta(2k+2q+2). \tag{3.81}$$

Proof.

$$\int_0^1 \frac{\ln^{2q}(x)\operatorname{Li}_{2p+1}(-x^2)}{1+x^2}\mathrm{d}x$$
$$= \int_0^\infty \frac{\ln^{2q}(x)\operatorname{Li}_{2p+1}(-x^2)}{1+x^2}\mathrm{d}x - \underbrace{\int_1^\infty \frac{\ln^{2q}(x)\operatorname{Li}_{2p+1}(-x^2)}{1+x^2}\mathrm{d}x}_{x\to 1/x}$$
$$= \int_0^\infty \frac{\ln^{2q}(x)\operatorname{Li}_{2p+1}(-x^2)}{1+x^2}\mathrm{d}x - \int_0^1 \frac{\ln^{2q}(x)\operatorname{Li}_{2p+1}(-\frac{1}{x^2})}{1+x^2}\mathrm{d}x$$
$$\{\text{add the integral to both sides then divide by 2}\}$$
$$= \frac{1}{2}\int_0^\infty \frac{\ln^{2q}(x)\operatorname{Li}_{2p+1}(-x^2)}{1+x^2}\mathrm{d}x$$
$$+\frac{1}{2}\int_0^1 \frac{\ln^{2q}(x)}{1+x^2}\left(\operatorname{Li}_{2p+1}(-x^2) - \operatorname{Li}_{2p+1}\left(-\frac{1}{x^2}\right)\right)\mathrm{d}x.$$

The first integral is given in (3.79). For the second, set $z = x^2$ in (1.95) to get

$$\int_0^1 \frac{\ln^{2q}(x)}{1+x^2}\left(\operatorname{Li}_{2p+1}(-x^2) - \operatorname{Li}_{2p+1}\left(-\frac{1}{x^2}\right)\right)\mathrm{d}x$$
$$= \int_0^1 \frac{\ln^{2q}(x)}{1+x^2}\left(-2\sum_{k=0}^{p}\frac{\eta(2p-2k)}{(2k+1)!}\ln^{2k+1}(x^2)\right)\mathrm{d}x$$

$$= -4\sum_{k=0}^{p}\frac{\eta(2p-2k)}{(2k+1)!}4^k\int_0^1\frac{\ln^{2k+2q+1}(x)}{1+x^2}\mathrm{d}x$$

$$\{\text{make use of (1.62) to get this integral}\}$$

$$= 4\sum_{k=0}^{p}\frac{(2k+2q+1)!}{(2k+1)!}4^k\eta(2p-2k)\beta(2k+2q+2).$$

Combining the two integral completes the proof.

Examples:

$$\int_0^1\frac{\ln^2(x)\,\mathrm{Li}_3(-x^2)}{1+x^2}\mathrm{d}x = 80\beta(6)+6\zeta(2)\beta(4)-\frac{93}{4}\pi\zeta(5)-\frac{3\pi^3}{8}\zeta(3);$$

$$\int_0^1\frac{\ln^4(x)\,\mathrm{Li}_3(-x^2)}{1+x^2}\mathrm{d}x = 3360\beta(8)+120\zeta(2)\beta(6)-\frac{5715}{8}\pi\zeta(7)$$
$$-\frac{279}{8}\pi^3\zeta(5)-\frac{15}{32}\pi^5\zeta(3);$$

$$\int_0^1\frac{\ln^4(x)\,\mathrm{Li}_5(-x^2)}{1+x^2}\mathrm{d}x = 48384\beta(10)+3360\zeta(2)\beta(8)+210\zeta(4)\beta(6)$$
$$-\frac{53655}{4}\pi\zeta(9)-\frac{5715}{16}\pi^3\zeta(7)-\frac{75}{32}\pi^5\zeta(5);$$

$$\int_0^1\frac{\ln^6(x)\,\mathrm{Li}_5(-x^2)}{1+x^2}\mathrm{d}x = 5322240\beta(12)+241920\zeta(2)\beta(10)+8820\zeta(4)\beta(8)$$
$$-\frac{9672075}{8}\pi\zeta(11)-\frac{804825}{16}\pi^3\zeta(9)-\frac{142875}{128}\pi^5\zeta(7)-\frac{915}{128}\pi^7\zeta(5).$$

3.1.48 $\int_0^1 \frac{\ln^{2q-1}(x)\,\mathrm{Li}_{2p}(-x^2)}{1+x^2}\mathrm{d}x$

Let $q, p \in \mathbb{Z}^+$. Then the following identity holds:

$$\int_0^1\frac{\ln^{2q-1}(x)\,\mathrm{Li}_{2p}(-x^2)}{1+x^2}\mathrm{d}x = \frac{-\left(\frac{\pi}{2}\right)^{2q-1}}{4(2p-1)!}*$$
$$\sum_{j=0}^{q-1}\binom{2q-1}{2j+1}\frac{(2p+2j)!}{\pi^{2j}}|E_{2q-2j-2}|(2^{2j+2p+1}-1)\zeta(2j+2p+1)$$
$$+\sum_{k=0}^{p}\frac{(2k+2q-1)!}{(2k)!}\,2^{2k}\eta(2p-2k)\beta(2k+2q). \tag{3.82}$$

Proof. We apply the previous proof,

$$\begin{aligned}
&\int_0^1 \frac{\ln^{2q-1}(x)\operatorname{Li}_{2p}(-x^2)}{1+x^2}\mathrm{d}x \\
&= \int_0^\infty \frac{\ln^{2q-1}(x)\operatorname{Li}_{2p}(-x^2)}{1+x^2}\mathrm{d}x - \underbrace{\int_1^\infty \frac{\ln^{2q-1}(x)\operatorname{Li}_{2p}(-x^2)}{1+x^2}\mathrm{d}x}_{x\to 1/x} \\
&= \int_0^\infty \frac{\ln^{2q-1}(x)\operatorname{Li}_{2p}(-x^2)}{1+x^2}\mathrm{d}x + \int_0^1 \frac{\ln^{2q-1}(x)\operatorname{Li}_{2p}(-\frac{1}{x^2})}{1+x^2}\mathrm{d}x \\
&= \frac{1}{2}\int_0^\infty \frac{\ln^{2q-1}(x)\operatorname{Li}_{2p}(-x^2)}{1+x^2}\mathrm{d}x \\
&+\frac{1}{2}\int_0^1 \frac{\ln^{2q-1}(x)}{1+x^2}\left(\operatorname{Li}_{2p}(-x^2)+\operatorname{Li}_{2p}\left(-\frac{1}{x^2}\right)\right)\mathrm{d}x.
\end{aligned}$$

The first integral is given in (3.80). For the second, set $z = x^2$ in (1.94)

$$\begin{aligned}
&\int_0^1 \frac{\ln^{2q-1}(x)}{1+x^2}\left(\operatorname{Li}_{2p}(-x^2)+\operatorname{Li}_{2p}\left(-\frac{1}{x^2}\right)\right)\mathrm{d}x \\
&= \int_0^1 \frac{\ln^{2q-1}(x)}{1+x^2}\left(-2\sum_{k=0}^{p}\frac{\eta(2p-2k)}{(2k)!}\ln^{2k}(x^2)\right)\mathrm{d}x \\
&= -2\sum_{k=0}^{p}\frac{\eta(2p-2k)}{(2k)!}4^k\int_0^1 \frac{\ln^{2k+2q-1}(x)}{1+x^2}\mathrm{d}x \\
&= 2\sum_{k=0}^{p}\frac{(2k+2q-1)!}{(2k)!}4^k\eta(2p-2k)\beta(2k+2q).
\end{aligned}$$

Put together the two integral to finish the proof.

Examples:

$$\int_0^1 \frac{\ln(x)\operatorname{Li}_2(-x^2)}{1+x^2}\mathrm{d}x = 6\beta(4)+\frac{1}{2}\zeta(2)\beta(2)-\frac{7\pi}{4}\zeta(3);$$

$$\begin{aligned}
\int_0^1 \frac{\ln^3(x)\operatorname{Li}_4(-x^2)}{1+x^2}\mathrm{d}x &= 1680\beta(8)+120\zeta(2)\beta(6)+\frac{21}{4}\zeta(4)\beta(4) \\
&-\frac{1905}{4}\pi\zeta(7)-\frac{93}{8}\pi^3\zeta(5);
\end{aligned}$$

$$\begin{aligned}
\int_0^1 \frac{\ln^5(x)\operatorname{Li}_4(-x^2)}{1+x^2}\mathrm{d}x &= 120960\beta(10)+5040\zeta(2)\beta(8)+105\zeta(4)\beta(6) \\
&-\frac{53655}{2}\pi\zeta(9)-\frac{9525}{8}\pi^3\zeta(7)-\frac{775}{32}\pi^5\zeta(5);
\end{aligned}$$

$$\int_0^1 \frac{\ln^5(x)\operatorname{Li}_6(-x^2)}{1+x^2}\mathrm{d}x = 1774080\beta(12) + 120960\zeta(2)\beta(10) + 8820\zeta(4)\beta(8) + \frac{465}{4}\zeta(6)\beta(6) - \frac{1934415}{4}\pi\zeta(11) - \frac{53655}{4}\pi^3\zeta(9) - \frac{9525}{64}\pi^5\zeta(7).$$

3.2 Integral Transformations

3.2.1 $\int_0^1 \frac{\ln^q(x)\ln^q(1-x)}{x(1-x)}\mathrm{d}x$

The following identity holds:

$$\int_0^1 \frac{\ln^q(x)\ln^q(1-x)}{x(1-x)}\mathrm{d}x = 2\int_0^1 \frac{\ln^q(x)\ln^q(1-x)}{x}\mathrm{d}x. \tag{3.83}$$

Proof.

$$\begin{aligned}
&\int_0^1 \frac{\ln^q(x)\ln^q(1-x)}{x(1-x)}\mathrm{d}x \\
&= \underbrace{\int_0^1 \frac{\ln^q(x)\ln^q(1-x)}{1-x}\mathrm{d}x}_{1-x\to x} + \int_0^1 \frac{\ln^q(x)\ln^q(1-x)}{x}\mathrm{d}x \\
&= \int_0^1 \frac{\ln^q(1-x)\ln^q(x)}{x}\mathrm{d}x + \int_0^1 \frac{\ln^q(x)\ln^q(1-x)}{x}\mathrm{d}x \\
&= 2\int_0^1 \frac{\ln^q(x)\ln^q(1-x)}{x}\mathrm{d}x.
\end{aligned}$$

3.2.2 $\int_0^{\frac{1}{2}} \frac{\ln^q(x)\ln^q(1-x)}{x(1-x)}\mathrm{d}x$

The following identity holds:

$$\int_0^{\frac{1}{2}} \frac{\ln^q(x)\ln^q(1-x)}{x(1-x)}\mathrm{d}x = \int_0^1 \frac{\ln^q(x)\ln^q(1-x)}{x}\mathrm{d}x. \tag{3.84}$$

Proof. Making the substitution $1-x \to x$,

$$\int_0^{\frac{1}{2}} \frac{\ln^q(1-x)\ln^q(x)}{x(1-x)}\mathrm{d}x = \int_{\frac{1}{2}}^1 \frac{\ln^q(x)\ln^q(1-x)}{(1-x)x}\mathrm{d}x$$

$$\left\{\text{add } \int_0^{\frac{1}{2}} \frac{\ln^q(1-x)\ln^q(x)}{x(1-x)}\mathrm{d}x \text{ to both sides then divide by } 2\right\}$$
$$= \frac{1}{2}\int_0^{\frac{1}{2}} \frac{\ln^q(1-x)\ln^q(x)}{x(1-x)}\mathrm{d}x + \frac{1}{2}\int_{\frac{1}{2}}^1 \frac{\ln^q(1-x)\ln^q(x)}{x(1-x)}\mathrm{d}x$$
$$= \frac{1}{2}\int_0^1 \frac{\ln^q(1-x)\ln^q(x)}{x(1-x)}\mathrm{d}x.$$

This integral is given in (3.83).

3.2.3 $\int_0^{\frac{1}{2}} \frac{\ln^q(2x)\ln(1-x)}{1-x}\mathrm{d}x$

The following identity holds:

$$\int_0^{\frac{1}{2}} \frac{\ln^q(2x)\ln(1-x)}{x(1-x)}\mathrm{d}x = (-1)^{q+1}q!\sum_{n=1}^{\infty}\frac{H_n}{2^n n^{q+1}}. \tag{3.85}$$

Proof. Start with subbing $x = \frac{y}{2}$,

$$\int_0^{\frac{1}{2}} \frac{\ln^q(2x)\ln(1-x)}{x(1-x)}\mathrm{d}x = \int_0^1 \frac{\ln^q(y)\ln(1-\frac{y}{2})}{y(1-\frac{y}{2})}\mathrm{d}y$$
$$\left\{\text{expand } \frac{\ln(1-y/2)}{1-y/2} \text{ in series as given in (2.4)}\right\}$$
$$= \int_0^1 \frac{\ln^q(y)}{y}\left(-\sum_{n=1}^{\infty}H_n\left(\frac{y}{2}\right)^n\right)\mathrm{d}y$$
$$= -\sum_{n=1}^{\infty}\frac{H_n}{2^n}\int_0^1 y^{n-1}\ln^q(y)\mathrm{d}y$$
$$= -\sum_{n=1}^{\infty}\frac{H_n}{2^n}\left(\frac{(-1)^q q!}{n^{q+1}}\right)$$
$$= (-1)^{q+1}q!\sum_{n=1}^{\infty}\frac{H_n}{2^n n^{q+1}}.$$

Following the same steps above, we obtain

$$\int_0^{\frac{1}{2}} \frac{\ln^q(2x)\ln(1-x)}{1-x}\mathrm{d}x = (-1)^{q+1}q!\sum_{n=1}^{\infty}\frac{H_{n-1}}{2^n n^{q+1}}$$

$$= (-1)^{q+1} q! \left(\sum_{n=1}^{\infty} \frac{H_n}{2^n n^{q+1}} - \sum_{n=1}^{\infty} \frac{1}{2^n n^{q+2}} \right)$$

$$= (-1)^q q! \left(\operatorname{Li}_{q+2}\left(\frac{1}{2}\right) - \sum_{n=1}^{\infty} \frac{H_n}{2^n n^{q+1}} \right). \tag{3.86}$$

3.2.4 $\int_0^1 \frac{\ln^q(1-x)\ln(1+x)}{x} \mathrm{d}x$

The following identity holds:

$$\int_0^1 \frac{\ln^q(1-x)\ln(1+x)}{x} \mathrm{d}x = (-1)^q q! \sum_{n=1}^{\infty} \frac{H_n^{(q+1)}}{n 2^n}. \tag{3.87}$$

Proof. Using the integral form of $H_n^{(q+1)}$ given in (1.122), we have

$$(-1)^q q! \sum_{n=1}^{\infty} \frac{H_n^{(q+1)}}{n 2^n} = \sum_{n=1}^{\infty} \frac{1}{n 2^n} \left(\int_0^1 \frac{\ln^q(x)(1-x^n)}{1-x} \mathrm{d}x \right)$$

$$= \int_0^1 \frac{\ln^q(x)}{1-x} \left(\sum_{n=1}^{\infty} \frac{1-x^n}{n 2^n} \right) \mathrm{d}x$$

$$= \int_0^1 \frac{\ln^q(x)}{1-x} \left(\sum_{n=1}^{\infty} \frac{1}{n 2^n} - \sum_{n=1}^{\infty} \frac{x^n}{n 2^n} \right) \mathrm{d}x$$

$$= \int_0^1 \frac{\ln^q(x)}{1-x} \left(\ln(2) + \ln\left(1 - \frac{x}{2}\right) \right) \mathrm{d}x$$

$$\overset{1-x \to x}{=} \int_0^1 \frac{\ln^q(1-x)\ln(1+x)}{x} \mathrm{d}x.$$

3.3 Results of Logarithmic Integrals

3.3.1 $\int_0^{\frac{\pi}{2}} \sin(2nx)\cot(x) \mathrm{d}x$

Let $n \in \mathbb{Z}^+$. Show that

$$\int_0^{\frac{\pi}{2}} \sin(2nx)\cot(x) \mathrm{d}x = \frac{\pi}{2}. \tag{3.88}$$

Solution. Since $\cos x = \Re\, e^{ix}$, which follows from Euler's formula, we have

$$\begin{aligned}\sum_{k=0}^{n-1}\cos((2k+1)x) &= \Re\sum_{k=0}^{n-1} e^{(2k+1)ix}\\ &= \Re\left\{e^{ix}\sum_{k=0}^{n-1}\left(e^{2ix}\right)^k\right\}\\ &\quad \{\text{use the geometric series formula}\}\\ &= \Re\left\{e^{ix}\frac{1-e^{2ixn}}{1-e^{2ix}}\right\}\\ &\quad \{\text{make use of } e^{ix}=\cos x + i\sin x \text{ and simplify}\}\\ &= \Re\left\{\frac{\sin(2nx)}{2\sin x}+i\frac{\sin^2(nx)}{\sin x}\right\}\\ &= \frac{\sin(2nx)}{2\sin x}.\end{aligned}$$

Separate the first term of the sum,

$$\frac{\sin(2nx)}{2\sin x} = \sum_{k=0}^{n-1}\cos((2k+1)x) = \cos x + \sum_{k=1}^{n-1}\cos((2k+1)x).$$

Multiply both sides by $2\cos x$,

$$\sin(2nx)\cot x = 2\cos^2 x + \sum_{k=1}^{n-1} 2\cos((2k+1)x)\cos x.$$

Use $2\cos^2 x = 1+\cos(2x)$ and $2\cos a\cos b = \cos(a-b)+\cos(a+b)$,

$$\sin(2nx)\cot x = 1+\cos(2x) + \sum_{k=1}^{n-1}\left[\cos(2kx)+\cos((2k+2)x)\right].$$

Now integrate both sides from $x=0$ to $\frac{\pi}{2}$,

$$\begin{aligned}&\int_0^{\frac{\pi}{2}}\sin(2nx)\cot x\,\mathrm{d}x\\ &= \int_0^{\frac{\pi}{2}}(1+\cos(2x))\mathrm{d}x + \int_0^{\frac{\pi}{2}}\sum_{k=1}^{n-1}[\cos(2kx)+\cos((2k+2)x)]\mathrm{d}x\\ &\quad \{\text{reverse the order of integration and summation}\}\end{aligned}$$

$$
\begin{aligned}
&= x + \frac{\sin(2x)}{2}\bigg|_0^{\frac{\pi}{2}} + \sum_{k=1}^{n-1}\int_0^{\frac{\pi}{2}}[\cos(2kx)+\cos((2k+2)x)]\mathrm{d}x \\
&= \frac{\pi}{2} + \frac{\sin(\pi)}{2} + \frac{1}{2}\sum_{k=1}^{n-1}\left[\frac{\sin(2kx)}{k}+\frac{\sin((2k+2)x)}{k+1}\right]_0^{\frac{\pi}{2}} \\
&= \frac{\pi}{2} + 0 + \frac{1}{2}\sum_{k=1}^{n-1}\frac{\sin(k\pi)}{k} + \frac{1}{2}\sum_{k=1}^{n-1}\frac{\sin(k\pi+\pi)}{k+1}
\end{aligned}
$$

$$\{\text{use } \sin(a+b)=\sin(a)\cos(b)+\cos(a)\sin(b) \text{ for the second sum}\}$$

$$= \frac{\pi}{2} + \frac{1}{2}\sum_{k=1}^{n-1}\frac{\sin(k\pi)}{k} + \frac{1}{2}\sum_{k=1}^{n-1}\frac{\sin(k\pi)\cos(\pi)+\cos(k\pi)\sin(\pi)}{k+1}$$

$$\{\text{note that } \sin(n\pi)=0 \text{ for integer } n\}$$

$$= \frac{\pi}{2}.$$

3.3.2 $\int_0^{\frac{\pi}{2}} \ln(\sin x)\mathrm{d}x$

Show that

$$\int_0^{\frac{\pi}{2}} \ln(\sin x)\mathrm{d}x = -\frac{\pi}{2}\ln(2). \tag{3.89}$$

Solution (i). By using the Fourier series of $\ln(\sin x)$ given in (2.90), we have

$$
\begin{aligned}
\int_0^{\frac{\pi}{2}} \ln(\sin x)\mathrm{d}x &= \int_0^{\frac{\pi}{2}}\left(-\ln(2)-\sum_{n=1}^{\infty}\frac{\cos(2nx)}{n}\right)\mathrm{d}x \\
&= -\int_0^{\frac{\pi}{2}}\ln(2)\mathrm{d}x - \sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\frac{\pi}{2}}\cos(2nx)\mathrm{d}x \\
&= -\frac{\pi}{2}\ln(2) - \sum_{n=1}^{\infty}\frac{1}{n}\cdot\frac{\sin(2nx)}{2n}\bigg|_0^{\frac{\pi}{2}} \\
&= -\frac{\pi}{2}\ln(2) - \sum_{n=1}^{\infty}\frac{1}{n}\cdot\frac{\sin(n\pi)}{2n} \\
&= -\frac{\pi}{2}\ln(2) - 0
\end{aligned}
$$

Solution (ii). Taking the log of both sides of the trigonometric identity:

$$\sin x = \frac{\sin(2x)}{2\cos x},$$

we get

$$\ln(\sin x) = \ln(\sin(2x)) - \ln(\cos x) - \ln(2).$$

Using this identity yields

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \ln(\sin x)\mathrm{d}x \\
&= \underbrace{\int_0^{\frac{\pi}{2}} \ln(\sin(2x))\mathrm{d}x}_{2x=y} - \underbrace{\int_0^{\frac{\pi}{2}} \ln(\cos x)\mathrm{d}x}_{x=\pi/2-y} - \int_0^{\frac{\pi}{2}} \ln(2)\mathrm{d}x \\
&= \frac{1}{2}\int_0^{\pi} \ln(\sin y)\mathrm{d}y - \int_0^{\frac{\pi}{2}} \ln\left(\cos\left(\frac{\pi}{2}-y\right)\right)\mathrm{d}y - \frac{\pi}{2}\ln(2) \\
&= \frac{1}{2}\left(\int_0^{\frac{\pi}{2}} \ln(\sin y)\mathrm{d}y + \underbrace{\int_{\frac{\pi}{2}}^{\pi} \ln(\sin y)\mathrm{d}y}_{y=\pi-u}\right) - \int_0^{\frac{\pi}{2}} \ln(\sin y)\mathrm{d}y - \frac{\pi}{2}\ln(2) \\
&= \frac{1}{2}\left(\int_0^{\frac{\pi}{2}} \ln(\sin y)\mathrm{d}y + \int_0^{\frac{\pi}{2}} \ln(\sin u)\mathrm{d}u\right) - \int_0^{\frac{\pi}{2}} \ln(\sin y)\mathrm{d}y - \frac{\pi}{2}\ln(2) \\
&= \int_0^{\frac{\pi}{2}} \ln(\sin y)\mathrm{d}y - \int_0^{\frac{\pi}{2}} \ln(\sin y)\mathrm{d}y - \frac{\pi}{2}\ln(2) \\
&= -\frac{\pi}{2}\ln(2).
\end{aligned}$$

Solution (iii).

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \ln(\sin x)\mathrm{d}x &= \lim_{b\to\frac{1}{2}} \int_0^{\frac{\pi}{2}} \sin^{2b-1}(x)\ln(\sin x)\mathrm{d}x \\
&= \lim_{b\to\frac{1}{2}} \int_0^{\frac{\pi}{2}} \frac{1}{2}\frac{\partial}{\partial b}\sin^{2b-1}(x)\mathrm{d}x \\
&= \frac{1}{2}\lim_{b\to\frac{1}{2}} \frac{d}{db}\int_0^{\frac{\pi}{2}} \sin^{2b-1}(x)\mathrm{d}x \\
&\quad \{\text{let } a = 1/2 \text{ in the beta function (1.49)}\} \\
&= \frac{1}{2}\lim_{b\to\frac{1}{2}} \frac{d}{db}\frac{1}{2}\,\mathrm{B}\left(\frac{1}{2}, b\right) \\
&= \frac{1}{4}\lim_{b\to\frac{1}{2}} \frac{d}{db}\frac{\Gamma(\frac{1}{2})\Gamma(b)}{\Gamma(b+\frac{1}{2})} \\
&\quad \{\text{use } \Gamma'(x) = \Gamma(x)\psi(x) \text{ given in (1.144)}\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\lim_{b\to\frac{1}{2}}\frac{\Gamma^2(\frac{1}{2})[\psi(b)-\psi(b+\frac{1}{2})]}{\Gamma(b+\frac{1}{2})}\\
&= \frac{1}{4}\frac{\Gamma^2(\frac{1}{2})[\psi(\frac{1}{2})-\psi(1)]}{\Gamma(1)}\\
&\{\text{write } \Gamma(1/2)=\sqrt{\pi} \text{ and } \psi(1)=-\gamma\}\\
&= \frac{\pi}{4}\left(\psi\left(\frac{1}{2}\right)+\gamma\right)\\
&\{\text{recall the result (1.148)}\}\\
&= \frac{\pi}{4}(-2\ln(2))\\
&= -\frac{\pi}{2}\ln(2).
\end{aligned}
$$

3.3.3 $\int_0^{\frac{\pi}{2}} x\ln(\sin x)\mathrm{d}x$

Show that

$$\int_0^{\frac{\pi}{2}} x\ln(\sin x)\mathrm{d}x = \frac{3}{16}\zeta(3)-\frac{3}{4}\ln(2)\zeta(2). \tag{3.90}$$

Solution. Again, by using the Fourier series of $\ln(\sin x)$, we get

$$
\begin{aligned}
\int_0^{\frac{\pi}{2}} x\ln(\sin x)\mathrm{d}x &= \int_0^{\frac{\pi}{2}} x\left(-\ln(2)-\sum_{n=1}^{\infty}\frac{\cos(2nx)}{n}\right)\mathrm{d}x\\
&= -\int_0^{\frac{\pi}{2}}\ln(2)x\,\mathrm{d}x-\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\frac{\pi}{2}} x\cos(2nx)\mathrm{d}x\\
&= -\frac{\pi^2}{8}\ln(2)-\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{\cos(2nx)}{4n^2}+\frac{x\sin(2nx)}{2n}\right]_0^{\frac{\pi}{2}}\\
&= -\frac{\pi^2}{8}\ln(2)-\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\cos(n\pi)}{4n^2}+\frac{\pi\sin(n\pi)}{4n}-\frac{1}{4n^2}\right)\\
&\left\{\text{use } \sum_{n=1}^{\infty}a_n\cos(n\pi)=\sum_{n=1}^{\infty}(-1)^n a_n \text{ given in (1.7)}\right\}\\
&= -\frac{\pi^2}{8}\ln(2)-\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{(-1)^n}{4n^2}+\frac{0}{4n}-\frac{1}{4n^2}\right)\\
&= -\frac{\pi^2}{8}\ln(2)-\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}+\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^3}
\end{aligned}
$$

$$= -\frac{\pi^2}{8}\ln(2) - \frac{1}{4}\operatorname{Li}_3(-1) + \frac{1}{4}\zeta(3).$$

The solution completes on writing $\operatorname{Li}_3(-1) = -\frac{3}{4}\zeta(3)$ and $\frac{\pi^2}{6} = \zeta(2)$. Moreover, by using the Fourier series of $\ln(\cos x)$ given in (2.93) and applying the same steps above, we get

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} x\ln(\cos x)\mathrm{d}x &= \int_0^{\frac{\pi}{2}} x\left(-\ln(2) - \sum_{n=1}^{\infty}\frac{(-1)^n\cos(2nx)}{n}\right)\mathrm{d}x \\
&= -\frac{3}{4}\ln(2)\zeta(2) - \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\frac{(-1)^n}{4n^2} + \frac{0}{4n} - \frac{1}{4n^2}\right) \\
&= -\frac{3}{4}\ln(2)\zeta(2) - \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^3} + \frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3} \\
&= -\frac{3}{4}\ln(2)\zeta(2) - \frac{7}{16}\zeta(3). \qquad (3.91)
\end{aligned}$$

3.3.4 $\int_0^{\frac{\pi}{2}} \ln^2(\sin x)\mathrm{d}x$

Show that

$$\int_0^{\frac{\pi}{2}} \ln^2(\sin x)\mathrm{d}x = \frac{\pi}{2}\ln^2(2) + \frac{\pi^3}{24}. \qquad (3.92)$$

Solution (i). Using

$$\ln^2(\sin x) = -\ln^2(2) - 2\ln(2)\ln(\sin x) + \ln^2(2\sin x),$$

we have

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \ln^2(\sin x)\mathrm{d}x \\
&= -\int_0^{\frac{\pi}{2}} \ln^2(2)\mathrm{d}x - 2\ln(2)\int_0^{\frac{\pi}{2}}\ln(\sin x)\mathrm{d}x + \int_0^{\frac{\pi}{2}}\ln^2(2\sin x)\mathrm{d}x.
\end{aligned}$$

The first integral is $\frac{\pi}{2}\ln^2(2)$ and the second integral is $-\frac{\pi}{2}\ln(2)$ given in (3.89). For the third one, use the Fourier series of $\ln^2(2\sin x)$ given in (2.110),

$$\int_0^{\frac{\pi}{2}} \ln^2(2\sin x)\mathrm{d}x = \int_0^{\frac{\pi}{2}}\left(\left(\frac{\pi}{2} - x\right)^2 + 2\sum_{n=1}^{\infty}\frac{H_{n-1}}{n}\cos(2nx)\right)\mathrm{d}x$$

$$
\begin{aligned}
&= \int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2}-x\right)^2 \mathrm{d}x + 2\sum_{n=1}^{\infty} \frac{H_{n-1}}{n} \int_0^{\frac{\pi}{2}} \cos(2nx)\mathrm{d}x \\
&= -\frac{1}{3}\left(\frac{\pi}{2}-x\right)^3 \Bigg|_0^{\frac{\pi}{2}} + 2\sum_{n=1}^{\infty} \frac{H_{n-1}}{n} \cdot \frac{\sin(2nx)}{2n}\Bigg|_0^{\frac{\pi}{2}} \\
&= \frac{\pi^3}{24} + 2\sum_{n=1}^{\infty} \frac{H_{n-1}}{n} \cdot \frac{\sin(n\pi)}{2n} \\
&= \frac{\pi^3}{24} + 0.
\end{aligned}
$$

Combining the three integrals completes the solution.

Solution (ii).

$$
\begin{aligned}
\int_0^{\frac{\pi}{2}} \ln^2(\sin x)\mathrm{d}x &= \lim_{b\to\frac{1}{2}} \int_0^{\frac{\pi}{2}} \sin^{2b-1}(x)\ln^2(\sin x)\mathrm{d}x \\
&= \lim_{b\to\frac{1}{2}} \int_0^{\frac{\pi}{2}} \frac{1}{4}\frac{\partial^2}{\partial b^2} \sin^{2b-1}(x)\mathrm{d}x \\
&= \frac{1}{4}\lim_{b\to\frac{1}{2}} \frac{d^2}{db^2} \int_0^{\frac{\pi}{2}} \sin^{2b-1}(x)\mathrm{d}x \\
&= \frac{1}{4}\lim_{b\to\frac{1}{2}} \frac{d^2}{db^2} \frac{1}{2}\mathrm{B}\left(\frac{1}{2}, b\right) \\
&= \frac{1}{8}\lim_{b\to\frac{1}{2}} \frac{d^2}{db^2} \frac{\Gamma(\frac{1}{2})\Gamma(b)}{\Gamma(b+\frac{1}{2})} \\
&= \frac{1}{8}\lim_{b\to\frac{1}{2}} \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\left[(\psi(b)-\psi(a+b))^2 + \psi^{(1)}(b) - \psi^{(1)}(a+b)\right] \\
&= \frac{1}{8}\frac{\Gamma^2(\frac{1}{2})}{\Gamma(1)}\left[\left(\psi\left(\frac{1}{2}\right)-\psi(1)\right)^2 + \psi^{(1)}\left(\frac{1}{2}\right) - \psi^{(1)}(1)\right]
\end{aligned}
$$

{recall the results (1.148), (1.157), and (1.155)}

$$
\begin{aligned}
&= \frac{\pi}{8}\left[(-2\ln(2))^2 + 3\zeta(2) - \zeta(2)\right] \\
&= \frac{\pi}{2}\ln^2(2) + \frac{\pi^3}{24}.
\end{aligned}
$$

3.3.5 $\int_0^{\frac{\pi}{2}} \ln(\sin x)\ln(\cos x)\mathrm{d}x$

Show that

$$\int_0^{\frac{\pi}{2}} \ln(\sin x)\ln(\cos x)\mathrm{d}x = \frac{\pi}{2}\ln^2(2) - \frac{\pi^3}{48}. \tag{3.93}$$

Solution (i). Let $a = \ln(\sin x)$ and $b = \ln(\cos x)$ in the algebraic identity:

$$ab = \frac{1}{2}a^2 + \frac{1}{2}b^2 - \frac{1}{2}(a-b)^2,$$

we have

$$\ln(\sin x)\ln(\cos x) = \frac{1}{2}\ln^2(\sin x) + \frac{1}{2}\ln^2(\cos x) - \frac{1}{2}\ln^2(\tan x).$$

Exploiting this identity gives

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \ln(\sin x)\ln(\cos x)\mathrm{d}x \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln^2(\sin x)\mathrm{d}x + \underbrace{\frac{1}{2}\int_0^{\frac{\pi}{2}} \ln^2(\cos x)\mathrm{d}x}_{x=\pi/2-y} - \underbrace{\frac{1}{2}\int_0^{\frac{\pi}{2}} \ln^2(\tan x)\mathrm{d}x}_{\tan x = y} \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln^2(\sin x)\mathrm{d}x + \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln^2(\sin y)\mathrm{d}y - \frac{1}{2}\int_0^{\infty} \frac{\ln^2(y)}{1+y^2}\mathrm{d}y \\
&= \int_0^{\frac{\pi}{2}} \ln^2(\sin x)\mathrm{d}x - \frac{1}{2}\int_0^{\infty} \frac{\ln^2(y)}{1+y^2}\mathrm{d}y.
\end{aligned}$$

These two integrals are given in (3.92) and (3.21).

Solution (ii).

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \ln(\sin x)\ln(\cos x)\mathrm{d}x \\
&= \lim_{\substack{a\to 1/2 \\ b\to 1/2}} \int_0^{\frac{\pi}{2}} \cos^{2a-1}(x)\sin^{2b-1}(x)\ln(\sin x)\ln(\cos x)\mathrm{d}x \\
&= \lim_{\substack{a\to 1/2 \\ b\to 1/2}} \int_0^{\frac{\pi}{2}} \frac{1}{4}\frac{\partial^2}{\partial a\partial b}\cos^{2a-1}(x)\sin^{2b-1}(x)\mathrm{d}x \\
&= \frac{1}{4}\lim_{\substack{a\to 1/2 \\ b\to 1/2}} \frac{\partial^2}{\partial a\partial b}\int_0^{\frac{\pi}{2}} \cos^{2a-1}(x)\sin^{2b-1}(x)\mathrm{d}x
\end{aligned}$$

$$= \frac{1}{4} \lim_{\substack{a \to 1/2 \\ b \to 1/2}} \frac{\partial^2}{\partial a \partial b} \frac{1}{2} \mathrm{B}(a,b)$$

$$= \frac{1}{8} \lim_{\substack{a \to 1/2 \\ b \to 1/2}} \frac{\partial^2}{\partial a \partial b} \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

$$= \frac{1}{8} \lim_{\substack{a \to 1/2 \\ b \to 1/2}} \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \Big[(\psi(a)-\psi(a+b))(\psi(b)-\psi(a+b)) - \psi^{(1)}(a+b)\Big]$$

$$= \frac{1}{8} \frac{\Gamma^2\left(\frac{1}{2}\right)}{\Gamma(1)} \left[\left(\psi\left(\frac{1}{2}\right) - \psi(1)\right)^2 - \psi^{(1)}(1)\right]$$

{recall the results (1.148) and (1.155)}

$$= \frac{\pi}{8} \left[(-2\ln(2))^2 - \frac{\pi^2}{6}\right]$$

$$= \frac{\pi}{2} \ln^2(2) - \frac{\pi^3}{48}.$$

3.3.6 $\int_0^{\frac{\pi}{2}} x \ln^2(\sin x) \mathrm{d}x$

Show that

$$\int_0^{\frac{\pi}{2}} x \ln^2(\sin x) \mathrm{d}x = \mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{19}{32}\zeta(4) + \frac{1}{2}\ln^2(2)\zeta(2) + \frac{1}{24}\ln^4(2). \quad (3.94)$$

Solution. Write $\ln^2(\sin x) = -\ln^2(2) - 2\ln(2)\ln(\sin x) + \ln^2(2\sin x)$,

$$\int_0^{\frac{\pi}{2}} x \ln^2(\sin x) \mathrm{d}x$$

$$= -\int_0^{\frac{\pi}{2}} x \ln^2(2) \mathrm{d}x - 2\ln(2) \int_0^{\frac{\pi}{2}} x \ln(\sin x) \mathrm{d}x + \int_0^{\frac{\pi}{2}} x \ln^2(2\sin x) \mathrm{d}x.$$

The first integral is $\frac{\pi^2}{8}\ln^2(2)$ and the second integral is given in (3.90). For the third one, again we use the Fourier series of $\ln^2(2\sin x)$ to get

$$\int_0^{\frac{\pi}{2}} x \ln^2(2\sin x) \mathrm{d}x$$

$$= \int_0^{\frac{\pi}{2}} x \left(\frac{\pi}{2} - x\right)^2 \mathrm{d}x + 2\sum_{n=1}^{\infty} \frac{H_{n-1}}{n} \int_0^{\frac{\pi}{2}} x \cos(2nx) \mathrm{d}x$$

$$= \frac{\pi^4}{192} + 2\sum_{n=1}^{\infty} \frac{H_{n-1}}{n} \left[\frac{\cos(2nx)}{4n^2} + \frac{x\sin(2nx)}{2n} - \frac{1}{4n^2} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{15}{32}\zeta(4) + 2\sum_{n=1}^{\infty} \frac{H_{n-1}}{n} \left(\frac{\cos(n\pi)}{4n^2} - \frac{\pi\sin(n\pi)}{4n} - \frac{1}{4n^2} \right)$$

$$= \frac{15}{32}\zeta(4) + 2\sum_{n=1}^{\infty} \frac{H_n - \frac{1}{n}}{n} \left(\frac{(-1)^n}{4n^2} - \frac{1}{4n^2} \right)$$

$$= \frac{15}{32}\zeta(4) - \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^4} - \frac{1}{2}\sum_{n=1}^{\infty} \frac{H_n}{n^3} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n^3}$$

$$\{\text{the second sum is } \operatorname{Li}_4(-1) = -7/8\,\zeta(4) \text{ and }\}$$

$$\{\text{the last two sums are calculated in (4.4) and (4.86) respectively}\}$$

$$= \operatorname{Li}_4\left(\frac{1}{2}\right) - \frac{19}{32}\zeta(4) + \frac{7}{8}\ln(2)\zeta(3) - \frac{1}{4}\ln^2(2)\zeta(2) + \frac{1}{24}\ln^4(2).$$

The solution finishes on grouping the three integrals.

3.3.7 $\int_0^{\frac{\pi}{2}} x \ln^2(\cos x) \mathrm{d}x$

Show that

$$\int_0^{\frac{\pi}{2}} x\ln^2(\cos x)\mathrm{d}x = -\operatorname{Li}_4\left(\frac{1}{2}\right) + \frac{79}{32}\zeta(4) + \ln^2(2)\zeta(2) - \frac{1}{24}\ln^4(2). \quad (3.95)$$

Solution. Write $\ln^2(\cos x) = -\ln^2(2) - 2\ln(2)\ln(\cos x) + \ln^2(2\cos x)$,

$$\int_0^{\frac{\pi}{2}} x\ln^2(\cos x)\mathrm{d}x$$

$$= -\int_0^{\frac{\pi}{2}} x\ln^2(2)\mathrm{d}x - 2\ln(2)\int_0^{\frac{\pi}{2}} x\ln(\cos x)\mathrm{d}x + \int_0^{\frac{\pi}{2}} x\ln^2(2\cos x)\mathrm{d}x.$$

The first integral is $\frac{\pi^2}{8}\ln^2(2)$ and the second integral is given in (3.91). For the third one, we use the Fourier series of $\ln^2(2\cos x)$ given in (2.109) to get

$$\int_0^{\frac{\pi}{2}} x\ln^2(2\cos x)\mathrm{d}x$$

$$= \int_0^{\frac{\pi}{2}} x\left(x^2 + 2\sum_{n=1}^{\infty} (-1)^n \frac{H_{n-1}}{n}\cos(2nx) \right)\mathrm{d}x$$

$$
\begin{aligned}
&= \int_0^{\frac{\pi}{2}} x^3 \mathrm{d}x + 2\sum_{n=1}^{\infty} \frac{(-1)^n H_{n-1}}{n} \int_0^{\frac{\pi}{2}} x\cos(2nx)\mathrm{d}x \\
&= \frac{\pi^4}{64} + 2\sum_{n=1}^{\infty} \frac{(-1)^n (H_n - \frac{1}{n})}{n} \left(\frac{(-1)^n}{4n^2} - \frac{1}{4n^2}\right) \\
&= \frac{45}{32}\zeta(4) - \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^4} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n}{n^3} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^3} \\
&= -\operatorname{Li}_4\left(\frac{1}{2}\right) + \frac{79}{32}\zeta(4) - \frac{7}{8}\ln(2)\zeta(3) + \frac{1}{4}\ln^2(2)\zeta(2) - \frac{1}{24}\ln^4(2).
\end{aligned}
$$

The solution finishes on grouping the three integrals.

3.3.8 $\int_0^1 \frac{\ln(1+x)}{1+x^2}\mathrm{d}x$

Show that

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}\mathrm{d}x = \frac{\pi}{8}\ln(2). \tag{3.96}$$

Solution. Let $x \to \frac{1-x}{1+x}$,

$$
\begin{aligned}
\int_0^1 \frac{\ln(1+x)}{1+x^2}\mathrm{d}x &= \int_0^1 \frac{\ln\left(\frac{2}{1+x}\right)}{1+x^2}\mathrm{d}x \\
&= \ln(2)\int_0^1 \frac{\mathrm{d}x}{1+x^2} - \int_0^1 \frac{\ln(1+x)}{1+x^2}\mathrm{d}x \\
&\{\text{add the integral to both sides then divide by } 2\} \\
&= \frac{1}{2}\ln(2)\int_0^1 \frac{\mathrm{d}x}{1+x^2} \\
&= \frac{\pi}{8}\ln(2).
\end{aligned}
$$

3.3.9 $\int_0^1 \frac{\ln(1-x)}{1+x^2}\mathrm{d}x$

Show that

$$\int_0^1 \frac{\ln(1-x)}{1+x^2}\mathrm{d}x = \frac{\pi}{8}\ln(2) - G. \tag{3.97}$$

Solution. Let $x \to \frac{1-x}{1+x}$,

$$\begin{aligned}\int_0^1 \frac{\ln(1-x)}{1+x^2}\mathrm{d}x &= \int_0^1 \frac{\ln\left(\frac{2x}{1+x}\right)}{1+x^2}\mathrm{d}x\\ &= \int_0^1 \frac{\ln(2x)}{1+x^2}\mathrm{d}x - \int_0^1 \frac{\ln(1+x)}{1+x^2}\mathrm{d}x\\ &\{\text{add the integral to both sides then divide by } 2\}\\ &= \frac{1}{2}\int_0^1 \frac{\ln(2x)}{1+x^2}\mathrm{d}x + \underbrace{\frac{1}{2}\int_0^1 \frac{\ln\left(\frac{1-x}{1+x}\right)}{1+x^2}\mathrm{d}x}_{x\to(1-x)/(1+x)}\\ &= \frac{1}{2}\ln(2)\int_0^1 \frac{\mathrm{d}x}{1+x^2} + \int_0^1 \frac{\ln(x)}{1+x^2}\mathrm{d}x.\end{aligned}$$

The first integral is $\frac{\pi}{4}$ and the second is $-G$ given in (1.163).

3.3.10 $\int_0^1 \frac{\ln(1-x)\ln(1+x)}{x}\mathrm{d}x$

Show that

$$\int_0^1 \frac{\ln(1-x)\ln(1+x)}{x}\mathrm{d}x = -\frac{5}{8}\zeta(3). \tag{3.98}$$

Solution. Using

$$\ln(1-x)\ln(1+x) = \frac{1}{4}\ln^2(1-x^2) - \frac{1}{4}\ln^2\left(\frac{1-x}{1+x}\right),$$

we have

$$\begin{aligned}\int_0^1 \frac{\ln(1-x)\ln(1+x)}{x}\mathrm{d}x &= \frac{1}{4}\underbrace{\int_0^1 \frac{\ln^2(1-x^2)}{x}\mathrm{d}x}_{1-x^2=y} - \frac{1}{4}\underbrace{\int_0^1 \frac{\ln^2\left(\frac{1-x}{1+x}\right)}{x}\mathrm{d}x}_{x=(1-y)/(1+y)}\\ &= \frac{1}{8}\int_0^1 \frac{\ln^2(y)}{1-y}\mathrm{d}y - \frac{1}{2}\int_0^1 \frac{\ln^2(y)}{1-y^2}\mathrm{d}y\\ &\left\{\text{write } \frac{1}{1-y^2} = \frac{1}{1-y} - \frac{y}{1-y^2} \text{ in the second integral}\right\}\end{aligned}$$

$$= -\frac{3}{8}\int_0^1 \frac{\ln^2(y)}{1-y}\mathrm{d}y + \frac{1}{2}\underbrace{\int_0^1 \frac{y\ln^2(y)}{1-y^2}\mathrm{d}y}_{y=\sqrt{t}}$$

$$= -\frac{3}{8}\int_0^1 \frac{\ln^2(y)}{1-y}\mathrm{d}y + \frac{1}{16}\int_0^1 \frac{\ln^2(t)}{1-t}\mathrm{d}t$$

$$= -\frac{5}{16}\int_0^1 \frac{\ln^2(y)}{1-y}\mathrm{d}y.$$

This integral is given in (3.8).

3.3.11 $\int_0^{\frac{1}{2}} \frac{\ln(x)\ln(1-x)}{1-x}\mathrm{d}x$

Show that

$$\int_0^{\frac{1}{2}} \frac{\ln(x)\ln(1-x)}{1-x}\mathrm{d}x = \frac{1}{8}\zeta(3) + \frac{1}{3}\ln^3(2). \tag{3.99}$$

Solution (i). Using

$$\ln(x)\ln(1-x) = \frac{1}{2}\ln^2(1-x) - \frac{1}{2}\ln^2\left(\frac{x}{1-x}\right) + \frac{1}{2}\ln^2(x),$$

we have

$$\int_0^{\frac{1}{2}} \frac{\ln(x)\ln(1-x)}{1-x}\mathrm{d}x$$

$$= \frac{1}{2}\int_0^{\frac{1}{2}} \frac{\ln^2(1-x)}{1-x}\mathrm{d}x - \frac{1}{2}\underbrace{\int_0^{\frac{1}{2}} \frac{\ln^2\left(\frac{x}{1-x}\right)}{1-x}\mathrm{d}x}_{x/(1-x)=y} + \frac{1}{2}\underbrace{\int_0^{\frac{1}{2}} \frac{\ln^2(x)}{1-x}\mathrm{d}x}_{\text{IBP}}$$

$$= \frac{1}{6}\ln^3(2) - \frac{1}{2}\int_0^1 \frac{\ln^2(y)}{1+y}\mathrm{d}y + \frac{1}{2}\ln^3(2) + \underbrace{\int_0^{\frac{1}{2}} \frac{\ln(x)\ln(1-x)}{x}\mathrm{d}x}_{\int_0^1 - \int_{1/2}^1}$$

$$= \frac{2}{3}\ln^3(2) - \frac{1}{2}\int_0^1 \frac{\ln^2(y)}{1+y}\mathrm{d}y$$

$$+ \underbrace{\int_0^1 \frac{\ln(x)\ln(1-x)}{x}\mathrm{d}x}_{\text{IBP}} - \underbrace{\int_{\frac{1}{2}}^1 \frac{\ln(x)\ln(1-x)}{x}\mathrm{d}x}_{1-x\to x}$$

$$= \frac{2}{3}\ln^3(2) - \frac{1}{2}\int_0^1 \frac{\ln^2(y)}{1+y}\mathrm{d}y + \frac{1}{2}\int_0^1 \frac{\ln^2(x)}{1-x}\mathrm{d}x - \int_0^{\frac{1}{2}} \frac{\ln(x)\ln(1-x)}{1-x}\mathrm{d}x$$

$$\{\text{add the integral to both sides then divide by } 2\}$$

$$= \frac{1}{3}\ln^3(2) + \frac{1}{2}\int_0^1 \frac{x\ln^2(x)}{1-x^2}\mathrm{d}x.$$

This integral is computed in (3.9).

Solution (ii). Apply integration by parts twice,

$$\int_0^{\frac{1}{2}} \frac{\ln(x)\ln(1-x)}{1-x}\mathrm{d}x = \operatorname{Li}_2(1-x)\ln(1-x)\Big|_0^{\frac{1}{2}} + \int_0^{\frac{1}{2}} \frac{\operatorname{Li}_2(1-x)}{1-x}\mathrm{d}x$$

$$= -\ln(2)\operatorname{Li}_2\left(\frac{1}{2}\right) - \operatorname{Li}_3(1-x)\Big|_0^{\frac{1}{2}}$$

$$= -\ln(2)\operatorname{Li}_2\left(\frac{1}{2}\right) - \operatorname{Li}_3\left(\frac{1}{2}\right) + \zeta(3).$$

The values of $\operatorname{Li}_2(\frac{1}{2})$ and $\operatorname{Li}_3(\frac{1}{2})$ are given in (1.81) and (1.92).

3.3.12 $\int_0^1 \frac{\ln(x)\ln(1+x)}{1+x}\mathrm{d}x$

Show that

$$\int_0^1 \frac{\ln(x)\ln(1+x)}{1+x}\mathrm{d}x = -\frac{1}{8}\zeta(3). \tag{3.100}$$

Solution. By writing

$$\ln^2(1+x) = \frac{1}{2}\ln^2(1-x^2) + \frac{1}{2}\ln^2\left(\frac{1-x}{1+x}\right) - \ln^2(1-x),$$

we have

$$\int_0^1 \frac{\ln^2(1+x)}{x}\mathrm{d}x$$

$$= \frac{1}{2}\underbrace{\int_0^1 \frac{\ln^2(1-x^2)}{x}\mathrm{d}x}_{1-x^2\to x} + \frac{1}{2}\underbrace{\int_0^1 \frac{\ln^2\left(\frac{1-x}{1+x}\right)}{x}\mathrm{d}x}_{(1-x)/(1+x)\to x} - \underbrace{\int_0^1 \frac{\ln^2(1-x)}{x}\mathrm{d}x}_{1-x\to x}$$

$$= \frac{1}{4}\int_0^1 \frac{\ln^2(x)}{1-x}\mathrm{d}x + \int_0^1 \frac{\ln^2(x)}{1-x^2}\mathrm{d}x - \int_0^1 \frac{\ln^2(x)}{1-x}\mathrm{d}x$$

$$\left\{\text{use } \frac{1}{1-x^2}-\frac{1}{1-x}=\frac{-x}{1-x^2}\right\}$$

$$=\frac{1}{4}\int_0^1\frac{\ln^2(x)}{1-x}\mathrm{d}x-\underbrace{\int_0^1\frac{x\ln^2(x)}{1-x^2}\mathrm{d}x}_{x^2\to x}$$

$$=\frac{1}{8}\int_0^1\frac{\ln^2(x)}{1-x}\mathrm{d}x=\frac{1}{4}\zeta(3). \tag{3.101}$$

The solution completes on applying integration by parts:

$$\int_0^1\frac{\ln(x)\ln(1+x)}{1+x}\mathrm{d}x=-\frac{1}{2}\int_0^1\frac{\ln^2(1+x)}{x}\mathrm{d}x.$$

3.3.13 $\int_0^1 \frac{\ln(x)\ln(1-x)}{x\sqrt{1-x}}\mathrm{d}x$

Show that

$$\int_0^1\frac{\ln(x)\ln(1-x)}{\sqrt{x}(1-x)}\mathrm{d}x=7\zeta(3)-6\ln(2)\zeta(2). \tag{3.102}$$

Solution (i). Make the change of variable $\sqrt{1-x}=y$,

$$\int_0^1\frac{\ln(x)\ln(1-x)}{\sqrt{x}(1-x)}\mathrm{d}x=4\int_0^1\frac{\ln(y)\ln(1-y^2)}{1-y^2}\mathrm{d}y$$

$$\overset{y=\frac{1-x}{1+x}}{=}2\int_0^1\frac{\ln\left(\frac{1-x}{1+x}\right)\ln\left(\frac{4x}{(1+x)^2}\right)}{x}\mathrm{d}x$$

$$=4\ln(2)\int_0^1\frac{\ln\left(\frac{1-x}{1+x}\right)}{x}\mathrm{d}x+2\int_0^1\frac{\ln(x)\ln\left(\frac{1-x}{1+x}\right)}{x}\mathrm{d}x+4\int_0^1\frac{\ln^2(1+x)}{x}\mathrm{d}x$$

$$-4\int_0^1\frac{\ln(1-x)\ln(1+x)}{x}\mathrm{d}x.$$

All these integrals are given in (3.15), (3.16), (3.101), and (3.98) respectively.

Solution (ii). Set $s=\frac{1}{2}$ in (3.39),

$$\int_0^1\frac{\ln(x)\ln(1-x)}{\sqrt{x}(1-x)}\mathrm{d}x=\psi^{(1)}\left(\frac{1}{2}\right)\left(\psi\left(\frac{1}{2}\right)+\gamma\right)-\frac{1}{2}\psi^{(2)}\left(\frac{1}{2}\right).$$

The values of these terms are given in (1.157) and (1.148).

3.3.14 $\int_0^1 \frac{\ln(x)\ln(1+x)}{1-x}\mathrm{d}x$

Show that

$$\int_0^1 \frac{\ln(x)\ln(1+x)}{1-x}\mathrm{d}x = \zeta(3) - \frac{3}{2}\ln(2)\zeta(2). \tag{3.103}$$

Solution. By integration by parts,

$$\begin{aligned}
\int_0^1 \frac{\ln(1-x)\ln(1+x)}{x}\mathrm{d}x &= -\ln(2)\zeta(2) + \int_0^1 \frac{\mathrm{Li}_2(x)}{1+x}\mathrm{d}x \\
&= -\ln(2)\zeta(2) + \int_0^1 \frac{1}{1+x}\left(\int_0^1 -\frac{x\ln(u)}{1-xu}\mathrm{d}u\right)\mathrm{d}x \\
&\qquad \{\text{change the order of integration}\} \\
&= -\ln(2)\zeta(2) + \int_0^1 \ln(u)\left(\int_0^1 \frac{-x}{(1+x)(1-ux)}\mathrm{d}x\right)\mathrm{d}u \\
&\qquad \{\text{evaluate the inner integral by partial fraction decomposition}\} \\
&= -\ln(2)\zeta(2) + \int_0^1 \ln(u)\left(\frac{\ln(2)}{1+u} + \frac{\ln(1-u)}{u} - \frac{\ln(1-u)}{1+u}\right)\mathrm{d}u \\
&= -\ln(2)\zeta(2) + \ln(2)\int_0^1 \frac{\ln(u)}{1+u}\mathrm{d}u + \underbrace{\int_0^1 \frac{\ln(u)\ln(1-u)}{u}\mathrm{d}u}_{\text{IBP}} \\
&\qquad - \underbrace{\int_0^1 \frac{\ln(u)\ln(1-u)}{1+u}\mathrm{d}u}_{\text{IBP}} \\
&= -\ln(2)\zeta(2) + \ln(2)\int_0^1 \frac{\ln(u)}{1+u}\mathrm{d}u + \frac{1}{2}\int_0^1 \frac{\ln^2(u)}{1-u}\mathrm{d}u \\
&\qquad - \int_0^1 \frac{\ln(u)\ln(1+u)}{1-u}\mathrm{d}u + \int_0^1 \frac{\ln(1-u)\ln(1+u)}{u}\mathrm{d}u \\
&\qquad \{\text{collect the results (3.11) and (3.8)}\} \\
&= -\ln(2)\zeta(2) - \frac{1}{2}\ln(2)\zeta(2) + \zeta(3) \\
&\qquad - \int_0^1 \frac{\ln(u)\ln(1+u)}{1-u}\mathrm{d}u + \int_0^1 \frac{\ln(1-u)\ln(1+u)}{u}\mathrm{d}u.
\end{aligned} \tag{3.104}$$

The solution completes on canceling $\int_0^1 \frac{\ln(1-x)\ln(1+x)}{x}\mathrm{d}x$ from both sides.

Moreover, plugging (3.98) in (3.104) gives

$$\int_0^1 \frac{\operatorname{Li}_2(x)}{1+x}\mathrm{d}x = \zeta(3) - \frac{1}{2}\ln(2)\zeta(2). \tag{3.105}$$

3.3.15 $\int_0^1 \frac{\ln(x)\ln(1-x)}{1+x}\mathrm{d}x$

Show that

$$\int_0^1 \frac{\ln(x)\ln(1-x)}{1+x}\mathrm{d}x = \frac{13}{8}\zeta(3) - \frac{3}{2}\ln(2)\zeta(2). \tag{3.106}$$

Solution. By integration by parts,

$$\begin{aligned}&\int_0^1 \frac{\ln(x)\ln(1-x)}{1+x}\mathrm{d}x\\ &= \int_0^1 \frac{\ln(x)\ln(1+x)}{1-x}\mathrm{d}x - \int_0^1 \frac{\ln(1-x)\ln(1+x)}{x}\mathrm{d}x.\end{aligned}$$

These two integrals are calculated in (3.103) and (3.98) respectively.

3.3.16 $\int_0^1 \frac{\ln(x)\arctan x}{1+x}\mathrm{d}x$

Show that

$$\int_0^1 \frac{\ln(x)\arctan x}{1+x}\mathrm{d}x = \frac{1}{2}\ln(2)G - \frac{\pi^3}{64}. \tag{3.107}$$

Solution. By integration by parts,

$$\begin{aligned}\int_0^1 \frac{\ln(x)\arctan x}{1+x}\mathrm{d}x &= \int_0^1 \mathrm{d}(\operatorname{Li}_2(-x) + \ln(x)\ln(1+x))\arctan x\\
&= (\operatorname{Li}_2(-x) + \ln(x)\ln(1+x))\arctan x\Big|_0^1 - \int_0^1 \frac{\operatorname{Li}_2(-x) + \ln(x)\ln(1+x)}{1+x^2}\mathrm{d}x\\
&= -\frac{\pi^3}{48} - \int_0^1 \frac{\operatorname{Li}_2(-x)}{1+x^2}\mathrm{d}x - \int_0^1 \frac{\ln(x)\ln(1+x)}{1+x^2}\mathrm{d}x\\
&\quad \{\text{recall the relation involving these two integrals from (3.127)}\}\\
&= -\frac{\pi^3}{48} - \frac{\pi}{16}\int_0^1 \frac{\ln(x)}{1+x}\mathrm{d}x - \frac{\ln(2)}{2}\int_0^1 \frac{\ln(y)}{1+y^2}\mathrm{d}y.\end{aligned}$$

These two integrals are evaluated in (3.11) and (1.163).

3.3.17 $\int_0^1 \frac{\ln(1-x)\ln(1+x)}{1+x^2}\mathrm{d}x$

Show that

$$\int_0^1 \frac{\ln(1-x)\ln(1+x)}{1+x^2}\mathrm{d}x = \Im\{\operatorname{Li}_3(1+i)\} - \frac{\pi^3}{32} - \ln(2)G. \tag{3.108}$$

Solution. By the substitution $x \to \frac{1-x}{1+x}$, we have

$$\int_0^1 \frac{\ln^2(x)}{1+x^2}\mathrm{d}x = \int_0^1 \frac{\ln^2\left(\frac{1-x}{1+x}\right)}{1+x^2}\mathrm{d}x$$

$$= \int_0^1 \frac{\ln^2(1-x)}{1+x^2}\mathrm{d}x - 2\int_0^1 \frac{\ln(1-x)\ln(1+x)}{1+x^2}\mathrm{d}x + \int_0^1 \frac{\ln^2(1+x)}{1+x^2}\mathrm{d}x$$

or

$$\int_0^1 \frac{\ln(1-x)\ln(1+x)}{1+x^2}\mathrm{d}x$$

$$= \frac{1}{2}\int_0^1 \frac{\ln^2(1-x)}{1+x^2}\mathrm{d}x + \frac{1}{2}\int_0^1 \frac{\ln^2(1+x)}{1+x^2}\mathrm{d}x - \frac{1}{2}\int_0^1 \frac{\ln^2(x)}{1+x^2}\mathrm{d}x.$$

These integrals are given in (3.26), (3.29), and (1.66) respectively.

3.3.18 $\int_0^1 \frac{\ln(x)\ln(1+x)}{1+x^2}\mathrm{d}x$

Show that

$$\int_0^1 \frac{\ln(x)\ln(1+x)}{1+x^2}\mathrm{d}x = 3\,\Im\{\operatorname{Li}_3(1+i)\} - \frac{5\pi^3}{64} - \frac{3\pi}{16}\ln^2(2) - 2\ln(2)G. \tag{3.109}$$

Solution.

$$\int_0^1 \frac{\ln^2(1+x)}{1+x^2}\mathrm{d}x = \int_0^\infty \frac{\ln^2(1+x)}{1+x^2}\mathrm{d}x - \underbrace{\int_1^\infty \frac{\ln^2(1+x)}{1+x^2}\mathrm{d}x}_{x\to 1/x}$$

$$= \int_0^\infty \frac{\ln^2(1+x)}{1+x^2}\mathrm{d}x - \int_0^1 \frac{\ln^2\left(\frac{1+x}{x}\right)}{1+x^2}\mathrm{d}x$$

$$= \int_0^\infty \frac{\ln^2(1+x)}{1+x^2}\mathrm{d}x - \int_0^1 \frac{\ln^2(1+x)}{1+x^2}\mathrm{d}x + 2\int_0^1 \frac{\ln(x)\ln(1+x)}{1+x^2}\mathrm{d}x$$

$$- \int_0^1 \frac{\ln^2(x)}{1+x^2}\mathrm{d}x.$$

or

$$\int_0^1 \frac{\ln(x)\ln(1+x)}{1+x^2}\mathrm{d}x$$

$$= \int_0^1 \frac{\ln^2(1+x)}{1+x^2}\mathrm{d}x - \frac{1}{2}\int_0^\infty \frac{\ln^2(1+x)}{1+x^2}\mathrm{d}x + \frac{1}{2}\int_0^1 \frac{\ln^2(x)}{1+x^2}\mathrm{d}x.$$

These integrals are given in (3.29), (3.23), and (1.66) respectively.

3.3.19 $\int_0^{\frac{1}{2}} \frac{\ln^2(x)\ln(1-x)}{1-x}\mathrm{d}x$

Show that

$$\int_0^{\frac{1}{2}} \frac{\ln^2(x)\ln(1-x)}{1-x}\mathrm{d}x = -\frac{1}{4}\ln^4(2) - \frac{1}{4}\zeta(4). \tag{3.110}$$

Solution. By integration by parts,

$$\int_0^{\frac{1}{2}} \frac{\ln^2(x)\ln(1-x)}{1-x}\mathrm{d}x = -\frac{1}{2}\ln^4(2) + \int_0^{\frac{1}{2}} \frac{\ln^2(1-x)\ln(x)}{x}\mathrm{d}x$$

$$\overset{1-x\to x}{=} -\frac{1}{2}\ln^4(2) + \int_{\frac{1}{2}}^1 \frac{\ln^2(x)\ln(1-x)}{1-x}\mathrm{d}x$$

$$\{\text{add the integral to both sides then divide by } 2\}$$

$$= -\frac{1}{4}\ln^4(2) + \frac{1}{2}\int_0^1 \frac{\ln^2(x)\ln(1-x)}{1-x}\mathrm{d}x$$

and this integral is given in (3.41).

3.3.20 $\int_0^1 \frac{\ln^2(x)\ln(1+x)}{1+x}\mathrm{d}x$

Show that

$$\begin{aligned}\int_0^1 \frac{\ln^2(x)\ln(1+x)}{1+x}\mathrm{d}x &= 4\operatorname{Li}_4\left(\frac{1}{2}\right) - \frac{15}{4}\zeta(4) + \frac{7}{2}\ln(2)\zeta(3) \\ &\quad - \ln^2(2)\zeta(2) + \frac{1}{6}\ln^4(2). \end{aligned} \tag{3.111}$$

Solution. By making the change of variable $y = \frac{x}{1+x}$,

$$\begin{aligned}
\int_0^1 \frac{\ln^2(x)\ln(1+x)}{1+x}\mathrm{d}x &= -\int_0^{\frac12} \frac{\ln^2\left(\frac{y}{1-y}\right)\ln(1-y)}{1-y}\mathrm{d}y \\
&= \underbrace{2\int_0^{\frac12} \frac{\ln(y)\ln^2(1-y)}{1-y}\mathrm{d}y}_{\text{IBP}} - \int_0^{\frac12} \frac{\ln^2(y)\ln(1-y)}{1-y}\mathrm{d}y - \int_0^{\frac12} \frac{\ln^3(1-y)}{1-y}\mathrm{d}y \\
&= -\frac{2}{3}\ln^4(2) + \frac{2}{3}\underbrace{\int_0^{\frac12} \frac{\ln^3(1-y)}{y}\mathrm{d}y}_{1-y=x} - \int_0^{\frac12} \frac{\ln^2(y)\ln(1-y)}{1-y}\mathrm{d}y + \frac{1}{4}\ln^4(2) \\
&= -\frac{5}{12}\ln^4(2) + \frac{2}{3}\underbrace{\int_{\frac12}^1 \frac{\ln^3(x)}{1-x}\mathrm{d}x}_{\int_0^1 - \int_0^{1/2}} - \int_0^{\frac12} \frac{\ln^2(y)\ln(1-y)}{1-y}\mathrm{d}y \\
&= -\frac{5}{12}\ln^4(2) + \frac{2}{3}\int_0^1 \frac{\ln^3(x)}{1-x}\mathrm{d}x - \frac{2}{3}\int_0^{\frac12} \frac{\ln^3(x)}{1-x}\mathrm{d}x - \int_0^{\frac12} \frac{\ln^2(y)\ln(1-y)}{1-y}\mathrm{d}y.
\end{aligned}$$

These integrals are found in (3.8), (3.19), and (3.110) respectively.

3.3.21 $\int_0^1 \frac{\ln(x)\ln(1-x)\ln(1+x)}{x}\mathrm{d}x$

Show that

$$\begin{aligned}\int_0^1 \frac{\ln(x)\ln(1-x)\ln(1+x)}{x}\mathrm{d}x &= 2\operatorname{Li}_4\left(\frac{1}{2}\right) - \frac{27}{16}\zeta(4) + \frac{7}{4}\ln(2)\zeta(3) \\ &\quad -\frac{1}{2}\ln^2(2)\zeta(2) + \frac{1}{12}\ln^4(2). \end{aligned} \tag{3.112}$$

Solution. By writing

$$\ln(1-x)\ln(1+x) = \frac{1}{2}\ln^2(1-x^2) - \frac{1}{2}\ln^2(1-x) - \frac{1}{2}\ln^2(1+x),$$

we have

$$\begin{aligned}
\int_0^1 \frac{\ln(x)\ln(1-x)\ln(1+x)}{x}\mathrm{d}x &= \frac{1}{2}\underbrace{\int_0^1 \frac{\ln(x)\ln^2(1-x^2)}{x}\mathrm{d}x}_{1-x^2=y} \\
&\quad -\frac{1}{2}\underbrace{\int_0^1 \frac{\ln(x)\ln^2(1-x)}{x}\mathrm{d}x}_{1-x=y} - \frac{1}{2}\underbrace{\int_0^1 \frac{\ln(x)\ln^2(1+x)}{x}\mathrm{d}x}_{\text{IBP}} \\
&= \frac{1}{8}\int_0^1 \frac{\ln(1-y)\ln^2(y)}{1-y}\mathrm{d}y - \frac{1}{2}\int_0^1 \frac{\ln(1-y)\ln^2(y)}{1-y}\mathrm{d}y \\
&\quad -\frac{1}{4}\ln^2(x)\ln^2(1+x)\Big|_0^1 + \frac{1}{2}\int_0^1 \frac{\ln^2(x)\ln(1+x)}{1+x}\mathrm{d}x \\
&= -\frac{3}{8}\int_0^1 \frac{\ln(1-y)\ln^2(y)}{1-y}\mathrm{d}y + \frac{1}{2}\int_0^1 \frac{\ln^2(x)\ln(1+x)}{1+x}\mathrm{d}x.
\end{aligned}$$

The solution finalizes on gathering the results (3.41) and (3.111).

3.3.22 $\int_0^1 \frac{\ln^2(x)\ln(1-x)}{1+x}\mathrm{d}x$

Show that

$$\int_0^1 \frac{\ln^2(x)\ln(1-x)}{1+x}\mathrm{d}x = -4\operatorname{Li}_4\left(\frac{1}{2}\right) + \zeta(4) + \ln^2(2)\zeta(2) - \frac{1}{6}\ln^4(2). \tag{3.113}$$

Solution. Force integration by parts,

$$\begin{aligned}
&\int_0^1 \frac{\ln^2(x)\ln(1-x)}{1+x}\mathrm{d}x \\
&= \int_0^1 \frac{\ln^2(x)\ln(1+x)}{1-x}\mathrm{d}x - 2\int_0^1 \frac{\ln(x)\ln(1-x)\ln(1+x)}{x}\mathrm{d}x.
\end{aligned}$$

These two integrals are given in (3.46) and (3.112) respectively.

3.3.23 $\int_0^1 \frac{\ln(x)\ln^2(1-x)}{1+x}\mathrm{d}x$

Show that

$$\int_0^1 \frac{\ln(x)\ln^2(1-x)}{1+x}\mathrm{d}x = -6\operatorname{Li}_4\left(\frac{1}{2}\right)+\frac{11}{4}\zeta(4)-\frac{1}{4}\ln^4(2). \qquad (3.114)$$

Solution. Employing

$$\ln(x)\ln^2(1-x)$$
$$= \ln^2(x)\ln(1-x)+\frac{1}{3}\ln^3\left(\frac{x}{1-x}\right)-\frac{1}{3}\ln^3(x)+\frac{1}{3}\ln^3(1-x)$$

gives

$$\int_0^1 \frac{\ln(x)\ln^2(1-x)}{1+x}\mathrm{d}x = \int_0^1 \frac{\ln^2(x)\ln(1-x)}{1+x}\mathrm{d}x$$
$$+\frac{1}{3}\int_0^1 \frac{\ln^3\left(\frac{x}{1-x}\right)}{1+x}\mathrm{d}x-\frac{1}{3}\int_0^1 \frac{\ln^3(x)}{1+x}\mathrm{d}x+\frac{1}{3}\int_0^1 \frac{\ln^3(1-x)}{1+x}\mathrm{d}x.$$

These integrals are given in (3.113), (3.31), (3.11), and (3.17) respectively.

3.3.24 $\int_0^1 \frac{\ln(1-x)\ln^2(1+x)}{x}\mathrm{d}x$

Show that

$$\int_0^1 \frac{\ln(1-x)\ln^2(1+x)}{x}\mathrm{d}x = -\frac{3}{8}\zeta(4). \qquad (3.115)$$

Solution. By writing

$$\ln(1-x)\ln^2(1+x) = \frac{1}{6}\ln^3(1-x^2)+\frac{1}{6}\ln^3\left(\frac{1-x}{1+x}\right)-\frac{1}{3}\ln^3(1-x),$$

we get

$$\int_0^1 \frac{\ln(1-x)\ln^2(1+x)}{x}\mathrm{d}x$$
$$= \frac{1}{6}\underbrace{\int_0^1 \frac{\ln^3(1-x^2)}{x}\mathrm{d}x}_{1-x^2=y}+\frac{1}{6}\int_0^1 \frac{\ln^3\left(\frac{1-x}{1+x}\right)}{x}\mathrm{d}x-\frac{1}{3}\underbrace{\int_0^1 \frac{\ln^3(1-x)}{x}\mathrm{d}x}_{1-x=y}$$

$$= \frac{1}{12}\int_0^1 \frac{\ln^3(y)}{1-y}\mathrm{d}y + \frac{1}{6}\int_0^1 \frac{\ln^3\left(\frac{1-x}{1+x}\right)}{x}\mathrm{d}x - \frac{1}{3}\int_0^1 \frac{\ln^3(y)}{1-y}\mathrm{d}y$$

$$= -\frac{1}{6}\int_0^1 \frac{\ln^3(y)}{1-y}\mathrm{d}y + \frac{1}{4}\int_0^1 \frac{\ln^3\left(\frac{1-x}{1+x}\right)}{x}\mathrm{d}x.$$

These two integrals are evaluated in (3.8) and (3.15) respectively.

3.3.25 $\int_0^1 \frac{\ln^2(1-x)\ln(1+x)}{x}\mathrm{d}x$

Show that

$$\int_0^1 \frac{\ln^2(1-x)\ln(1+x)}{x}\mathrm{d}x = 2\operatorname{Li}_4\left(\frac{1}{2}\right) - \frac{5}{8}\zeta(4) + \frac{7}{4}\ln(2)\zeta(3) - \frac{1}{2}\ln^2(2)\zeta(2) + \frac{1}{12}\ln^4(2). \tag{3.116}$$

Solution. By writing

$$\ln^2(1-x)\ln(1+x) = \frac{1}{6}\ln^3(1-x^2) - \frac{1}{6}\ln^3\left(\frac{1-x}{1+x}\right) - \frac{1}{3}\ln^3(1+x),$$

we get

$$\int_0^1 \frac{\ln^2(1-x)\ln(1+x)}{x}\mathrm{d}x$$

$$= \frac{1}{6}\underbrace{\int_0^1 \frac{\ln^3(1-x^2)}{x}\mathrm{d}x}_{1-x^2=y} - \frac{1}{6}\int_0^1 \frac{\ln^3\left(\frac{1-x}{1+x}\right)}{x}\mathrm{d}x - \frac{1}{3}\int_0^1 \frac{\ln^3(1+x)}{x}\mathrm{d}x$$

$$= \frac{1}{12}\int_0^1 \frac{\ln^3(y)}{1-y}\mathrm{d}y - \frac{1}{6}\int_0^1 \frac{\ln^3\left(\frac{1-x}{1+x}\right)}{x}\mathrm{d}x - \frac{1}{3}\int_0^1 \frac{\ln^3(1+x)}{x}\mathrm{d}x.$$

These three integrals are given in (3.8), (3.15), and (3.20).

3.3.26 $\int_0^1 \frac{\ln(x)\ln^2(1-x)}{\sqrt{x}(1-x)}\mathrm{d}x$

Show that

$$\int_0^1 \frac{\ln(x)\ln^2(1-x)}{\sqrt{x}(1-x)}\mathrm{d}x = -15\zeta(4)+28\ln(2)\zeta(3)-12\ln^2(2)\zeta(2). \quad (3.117)$$

Solution. Make the change of variable $\sqrt{1-x}=y$,

$$\begin{aligned}
\int_0^1 \frac{\ln(x)\ln^2(1-x)}{\sqrt{x}(1-x)}\mathrm{d}x &= 4\int_0^1 \frac{\ln(y)\ln^2(1-y^2)}{1-y^2}\mathrm{d}y \\
&\overset{y=\frac{1-x}{1+x}}{=} 2\int_0^1 \frac{\ln\left(\frac{1-x}{1+x}\right)\ln^2\left(\frac{4x}{(1+x)^2}\right)}{x}\mathrm{d}x \\
= 8\ln^2(2)\int_0^1 \frac{\ln\left(\frac{1-x}{1+x}\right)}{x}\mathrm{d}x &+ 8\ln(2)\int_0^1 \frac{\ln\left(\frac{1-x}{1+x}\right)\ln(x)}{x}\mathrm{d}x \\
-8\int_0^1 \frac{\ln(x)\ln(1-x)\ln(1+x)}{x}\mathrm{d}x &+ 8\int_0^1 \frac{\ln(x)\ln^2(1+x)}{x}\mathrm{d}x \\
-16\ln(2)\int_0^1 \frac{\ln(1-x)\ln(1+x)}{x}\mathrm{d}x &+ 16\ln(2)\int_0^1 \frac{\ln^2(1+x)}{x}\mathrm{d}x \\
+8\int_0^1 \frac{\ln(1-x)\ln^2(1+x)}{x}\mathrm{d}x &- 8\int_0^1 \frac{\ln^3(1+x)}{x}\mathrm{d}x.
\end{aligned}$$

All these integrals, but the fourth, are given in (3.15), (3.16), (3.112), (3.98), (3.20), (3.115), and (3.20). Regarding the fourth integral, integrate it by parts,

$$\int_0^1 \frac{\ln(x)\ln^2(1+x)}{x}\mathrm{d}x = -\int_0^1 \frac{\ln^2(x)\ln(1+x)}{1+x}\mathrm{d}x,$$

which is given in (3.111).

A different method is by differentiating the beta function in (1.46),

$$\int_0^1 \frac{\ln(x)\ln^2(1-x)}{\sqrt{x}(1-x)}\mathrm{d}x = \lim_{\substack{a\to 1/2\\ b\to 0}} \frac{\partial^3}{\partial a\partial b^2}\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

but lengthy calculations will be involved and we had better let *Mathematica* do it.

The *Mathematica* command for $\lim\limits_{\substack{a\to 1/2\\ b\to 0}} \frac{\partial^3}{\partial a\partial b^2}\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is

```
Normal[Series[D[Gamma[a]Gamma[b]/Gamma[a+b],{a,1},{b,2}]
,{a,1/2,0},{b,0,0}]]//FullSimplify//Expand
```

3.3.27 $\int_0^1 \frac{\ln^2(x)\arctan x}{x(1+x^2)}\mathrm{d}x$

Show that

$$\int_0^1 \frac{\ln^2(x)\arctan x}{x(1+x^2)}\mathrm{d}x = \frac{\pi^3}{16}\ln(2) - \frac{7\pi}{64}\zeta(3) + \beta(4). \tag{3.118}$$

The following solution may be found in [5]:

Solution. Let I denote our integral,

$$I = \int_0^\infty \frac{\ln^2(x)\arctan x}{x(1+x^2)}\mathrm{d}x - \underbrace{\int_1^\infty \frac{\ln^2(x)\arctan x}{x(1+x^2)}\mathrm{d}x}_{x\to 1/x}$$

$$= \int_0^\infty \frac{\ln^2(x)\arctan x}{x(1+x^2)}\mathrm{d}x - \int_0^1 \frac{x\ln^2(x)\arctan(1/x)}{1+x^2}\mathrm{d}x$$

$$\left\{\text{write } \arctan(1/x) = \frac{\pi}{2} - \arctan x\right\}$$

$$= \int_0^\infty \frac{\ln^2(x)\arctan x}{x(1+x^2)}\mathrm{d}x - \frac{\pi}{2}\int_0^1 \frac{x\ln^2(x)}{1+x^2}\mathrm{d}x + \int_0^1 \frac{x\ln^2(x)\arctan x}{1+x^2}\mathrm{d}x$$

$$\left\{\text{let } x^2 \to x \text{ in the second integral}\right\}$$

$$\left\{\text{and write } \frac{x}{1+x^2} = \frac{1}{x} - \frac{1}{x(1+x^2)} \text{ in the third one}\right\}$$

$$= \int_0^\infty \frac{\ln^2(x)\arctan x}{x(1+x^2)}\mathrm{d}x - \frac{\pi}{16}\int_0^1 \frac{\ln^2(x)}{1+x}\mathrm{d}x + \int_0^1 \frac{\ln^2(x)\arctan x}{x}\mathrm{d}x - I$$

$$\{\text{add } I \text{ to both sides and integrate the third integral by parts then divide by } 2\}$$

$$= \frac{1}{2}\int_0^\infty \frac{\ln^2(x)\arctan x}{x(1+x^2)}\mathrm{d}x - \frac{\pi}{32}\int_0^1 \frac{\ln^2(x)}{1+x}\mathrm{d}x - \frac{1}{6}\int_0^1 \frac{\ln^3(x)}{1+x^2}\mathrm{d}x$$

$$\{\text{recall the results (3.11) and (1.62)}\}$$

$$= \frac{1}{2}\int_0^\infty \frac{\ln^2(x)\arctan x}{x(1+x^2)}\mathrm{d}x - \frac{3\pi}{64}\zeta(3) + \beta(4).$$

For the remaining integral, write $\arctan x = \int_0^1 \frac{x}{1+x^2y^2}\mathrm{d}y$,

$$\int_0^\infty \frac{\ln^2(x)\arctan x}{x(1+x^2)}\mathrm{d}x = \int_0^\infty \frac{\ln^2(x)}{x(1+x^2)}\left(\int_0^1 \frac{x}{1+x^2y^2}\mathrm{d}y\right)\mathrm{d}x$$

$$\left\{\text{write } \frac{1}{(1+x^2)(1+x^2y^2)} = \frac{1}{1-y^2}\left(\frac{1}{1+x^2} - \frac{y^2}{1+x^2y^2}\right)\right\}$$

$$\{\text{then change the order of integration}\}$$

$$= \int_0^1 \frac{1}{1-y^2}\left(\underbrace{\int_0^\infty \frac{\ln^2(x)}{1+x^2}\mathrm{d}x}_{x=t} - \underbrace{\int_0^\infty \frac{y^2\ln^2(x)}{1+x^2y^2}\mathrm{d}x}_{xy=t}\right)\mathrm{d}y$$

$$= \int_0^1 \frac{1}{1-y^2}\left(\int_0^\infty \frac{\ln^2(t)}{1+t^2}\mathrm{d}t - \int_0^\infty \frac{y\ln^2(\frac{t}{y})}{1+t^2}\mathrm{d}t\right)\mathrm{d}y$$

$$\{\text{write } \ln^2(t/y) = \ln^2(t) - 2\ln(y)\ln(t) + \ln^2(y)\}$$

$$= \int_0^1 \frac{1}{1-y^2}\left((1-y)\int_0^\infty \frac{\ln^2(t)}{1+t^2}\mathrm{d}t + 2y\ln(y)\int_0^\infty \frac{\ln(t)}{1+t^2}\mathrm{d}t \right.$$

$$\left. -y\ln^2(y)\int_0^\infty \frac{\mathrm{d}t}{1+t^2}\right)\mathrm{d}y$$

$$\{\text{recall the result of the first integral from (3.21)}\}$$

$$\{\text{and note that the second integral is 0 by using (3.22)}\}$$

$$= \int_0^1 \frac{1}{1-y^2}\left((1-y)\frac{\pi^3}{8} - \frac{\pi y\ln^2(y)}{2}\right)\mathrm{d}y$$

$$= \frac{\pi^3}{8}\int_0^1 \frac{1-y}{1-y^2}\mathrm{d}y - \frac{\pi}{2}\underbrace{\int_0^1 \frac{y\ln^2(y)}{1-y^2}\mathrm{d}y}_{y^2=x}$$

$$= \frac{\pi^3}{8}\int_0^1 \frac{\mathrm{d}y}{1+y} - \frac{\pi}{16}\int_0^1 \frac{\ln^2(x)}{1-x}\mathrm{d}x$$

$$\{\text{the second integral is given in (3.8)}\}$$

$$= \frac{\pi^3}{8}\ln(2) - \frac{\pi}{8}\zeta(3).$$

3.3.28 $\int_0^{\frac{1}{2}} \frac{\ln^3(x)\ln(1-x)}{1-x}\mathrm{d}x$

Show that

$$\int_0^{\frac{1}{2}} \frac{\ln^3(x)\ln(1-x)}{1-x}\mathrm{d}x = 6\operatorname{Li}_5\left(\frac{1}{2}\right) + 6\ln(2)\operatorname{Li}_4\left(\frac{1}{2}\right) + \frac{3}{16}\zeta(5)$$
$$-3\zeta(2)\zeta(3) + \frac{21}{8}\ln^2(2)\zeta(3) - \ln^3(2)\zeta(2) + \frac{9}{20}\ln^5(2). \qquad (3.119)$$

Solution. Make use of

$$\ln^3(x)\ln(1-x) = \frac{1}{4}\ln^4(x) + \frac{1}{4}\ln^4(1-x)$$

$$-\frac{1}{4}\ln^4\left(\frac{x}{1-x}\right)-\ln(x)\ln^3(1-x)+\frac{3}{2}\ln^2(x)\ln^2(1-x),$$

we have

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\frac{\ln^3(x)\ln(1-x)}{1-x}\mathrm{d}x\\
&=\frac{1}{4}\int_0^{\frac{1}{2}}\frac{\ln^4(x)}{1-x}\mathrm{d}x+\frac{1}{4}\int_0^{\frac{1}{2}}\frac{\ln^4(1-x)}{1-x}\mathrm{d}x-\frac{1}{4}\underbrace{\int_0^{\frac{1}{2}}\frac{\ln^4\left(\frac{x}{1-x}\right)}{1-x}\mathrm{d}x}_{x/(1-x)=y}\\
&-\underbrace{\int_0^{\frac{1}{2}}\frac{\ln(x)\ln^3(1-x)}{1-x}\mathrm{d}x}_{\text{IBP}}+\frac{3}{2}\underbrace{\int_0^{\frac{1}{2}}\frac{\ln^2(x)\ln^2(1-x)}{1-x}\mathrm{d}x}_{\text{IBP}}\\
&=\frac{1}{4}\int_0^{\frac{1}{2}}\frac{\ln^4(x)}{1-x}\mathrm{d}x+\frac{\ln^5(2)}{20}-\frac{1}{4}\int_0^1\frac{\ln^4(y)}{1+y}\mathrm{d}y\\
&-\frac{\ln^5(2)}{4}-\frac{1}{4}\underbrace{\int_0^{\frac{1}{2}}\frac{\ln^4(1-x)}{x}\mathrm{d}x}_{1-x\to x}+\frac{\ln^5(2)}{2}+\underbrace{\int_0^{\frac{1}{2}}\frac{\ln^3(1-x)\ln(x)}{x}\mathrm{d}x}_{1-x\to x}\\
&=\frac{3}{10}\ln^5(2)+\frac{1}{4}\int_0^{\frac{1}{2}}\frac{\ln^4(x)}{1-x}\mathrm{d}x-\frac{1}{4}\int_0^1\frac{\ln^4(y)}{1+y}\mathrm{d}y\\
&-\frac{1}{4}\underbrace{\int_{\frac{1}{2}}^1\frac{\ln^4(x)}{1-x}\mathrm{d}x}_{\int_0^1-\int_0^{1/2}}+\underbrace{\int_{\frac{1}{2}}^1\frac{\ln^3(x)\ln(1-x)}{1-x}\mathrm{d}x}_{\int_0^1-\int_0^{1/2}}\\
&=\frac{3}{10}\ln^5(2)+\frac{1}{2}\int_0^{\frac{1}{2}}\frac{\ln^4(x)}{1-x}\mathrm{d}x-\frac{1}{2}\int_0^1\frac{\ln^4(x)}{1-x^2}\mathrm{d}x\\
&+\int_0^1\frac{\ln^3(x)\ln(1-x)}{1-x}\mathrm{d}x-\int_0^{\frac{1}{2}}\frac{\ln^3(x)\ln(1-x)}{1-x}\mathrm{d}x\\
&\{\text{add the integral to both sides then divide by } 2\}\\
&=\frac{3\ln^5(2)}{20}+\frac{1}{4}\int_0^{\frac{1}{2}}\frac{\ln^4(x)}{1-x}\mathrm{d}x-\frac{1}{4}\int_0^1\frac{\ln^4(x)}{1-x^2}\mathrm{d}x+\frac{1}{2}\int_0^1\frac{\ln^3(x)\ln(1-x)}{1-x}\mathrm{d}x.
\end{aligned}\tag{3.120}$$

Gather the results (3.19), (3.14), and (3.41) to finish the solution.

3.3.29 $\int_0^1 \frac{\ln^3(x)\ln(1+x)}{1+x}\mathrm{d}x$

Show that

$$\int_0^1 \frac{\ln^3(x)\ln(1+x)}{1+x}\mathrm{d}x = \frac{87}{16}\zeta(5) - 3\zeta(2)\zeta(3). \tag{3.121}$$

Solution. Using $\frac{1}{1+x} = \frac{1}{x} - \frac{1}{x(1+x)}$, we have

$$\begin{aligned}
&\int_0^1 \frac{\ln^3(x)\ln(1+x)}{1+x}\mathrm{d}x \\
&= \underbrace{\int_0^1 \frac{\ln^3(x)\ln(1+x)}{x}\mathrm{d}x}_{\text{IBP}} - \int_0^1 \frac{\ln^3(x)\ln(1+x)}{x(1+x)}\mathrm{d}x \\
&= -\frac{1}{4}\int_0^1 \frac{\ln^4(x)}{1+x}\mathrm{d}x - \int_0^1 \frac{\ln^3(x)\ln(1+x)}{x(1+x)}\mathrm{d}x.
\end{aligned}$$

Putting $q = 2$ in (3.37) gives

$$\begin{aligned}
&\int_0^1 \frac{\ln^3(x)\ln(1+x)}{x(1+x)}\mathrm{d}x \\
&= \frac{1}{2}\int_0^\infty \frac{\ln^3(x)\ln(1+x)}{x(1+x)}\mathrm{d}x - \frac{5}{8}\int_0^1 \frac{\ln^4(x)}{1+x}\mathrm{d}x \\
&\left\{\text{let } x = \frac{1-t}{t} \text{ in the first integral}\right\} \\
&= \frac{1}{2}\int_0^1 \frac{\ln^3\left(\frac{t}{1-t}\right)\ln(t)}{1-t}\mathrm{d}t - \frac{5}{8}\int_0^1 \frac{\ln^4(x)}{1+x}\mathrm{d}x \\
&= \frac{1}{2}\int_0^1 \frac{\ln^4(t)}{1-t}\mathrm{d}t - \frac{3}{2}\int_0^1 \frac{\ln^3(t)\ln(1-t)}{1-t}\mathrm{d}t \\
&+\frac{3}{2}\underbrace{\int_0^1 \frac{\ln^2(t)\ln^2(1-t)}{1-t}\mathrm{d}t}_{\text{IBP}} - \frac{1}{2}\underbrace{\int_0^1 \frac{\ln(t)\ln^3(1-t)}{1-t}\mathrm{d}t}_{\text{IBP}} - \frac{5}{8}\int_0^1 \frac{\ln^4(x)}{1+x}\mathrm{d}x \\
&= \frac{1}{2}\int_0^1 \frac{\ln^4(t)}{1-t}\mathrm{d}t - \frac{3}{2}\int_0^1 \frac{\ln^3(t)\ln(1-t)}{1-t}\mathrm{d}t \\
&+\underbrace{\int_0^1 \frac{\ln^3(1-t)\ln(t)}{t}\mathrm{d}t}_{1-t\to t} - \frac{1}{8}\underbrace{\int_0^1 \frac{\ln^4(1-t)}{t}\mathrm{d}t}_{1-t\to t} - \frac{5}{8}\int_0^1 \frac{\ln^4(x)}{1+x}\mathrm{d}x
\end{aligned}$$

$$= \frac{3}{8}\int_0^1 \frac{\ln^4(t)}{1-t}\mathrm{d}t - \frac{1}{2}\int_0^1 \frac{\ln^3(t)\ln(1-t)}{1-t}\mathrm{d}t - \frac{5}{8}\int_0^1 \frac{\ln^4(x)}{1+x}\mathrm{d}x. \tag{3.122}$$

Put this integral back,

$$\begin{aligned}&\int_0^1 \frac{\ln^3(x)\ln(1+x)}{1+x}\mathrm{d}x \\ &= \frac{3}{8}\int_0^1 \frac{\ln^4(x)}{1+x}\mathrm{d}x - \frac{3}{8}\int_0^1 \frac{\ln^4(t)}{1-t}\mathrm{d}t + \frac{1}{2}\int_0^1 \frac{\ln^3(t)\ln(1-t)}{1-t}\mathrm{d}t \\ &= -\frac{3}{8}\underbrace{\int_0^1 \frac{2x\ln^4(x)}{1-x^2}\mathrm{d}x}_{x=\sqrt{t}} + \frac{1}{2}\int_0^1 \frac{\ln^3(t)\ln(1-t)}{1-t}\mathrm{d}t \\ &= -\frac{3}{128}\int_0^1 \frac{\ln^4(t)}{1-t}\mathrm{d}t + \frac{1}{2}\int_0^1 \frac{\ln^3(t)\ln(1-t)}{1-t}\mathrm{d}t.\end{aligned}$$

These two integrals are given in (3.8) and (3.41) respectively.

3.3.30 $\int_0^1 \frac{\ln^3(1-x)\ln(1+x)}{x}\mathrm{d}x$

Show that

$$\begin{aligned}\int_0^1 \frac{\ln^3(1-x)\ln(1+x)}{x}\mathrm{d}x &= 6\operatorname{Li}_5\left(\frac{1}{2}\right) + 6\ln(2)\operatorname{Li}_4\left(\frac{1}{2}\right) - \frac{81}{16}\zeta(5) \\ &-\frac{21}{8}\zeta(2)\zeta(3) + \frac{21}{8}\ln^2(2)\zeta(3) - \ln^3(2)\zeta(2) + \frac{1}{5}\ln^5(2).\end{aligned} \tag{3.123}$$

The following solution may be found in [21]:

Solution. Make use of

$$\begin{aligned}\ln^3(1-x)\ln(1+x) &= \frac{1}{4}\ln^4(1-x) - \frac{1}{4}\ln^4(1+x) \\ &+\frac{1}{16}\ln^4(1-x^2) - \frac{5}{16}\ln^4\left(\frac{1-x}{1+x}\right) - \frac{1}{2}\ln^3\left(\frac{1-x}{1+x}\right)\ln(1+x),\end{aligned}$$

we have

$$\begin{aligned}&\int_0^1 \frac{\ln^3(1-x)\ln(1+x)}{x}\mathrm{d}x \\ &= \frac{1}{4}\underbrace{\int_0^1 \frac{\ln^4(1-x)}{x}\mathrm{d}x}_{1-x=y} - \frac{1}{4}\int_0^1 \frac{\ln^4(1+x)}{x}\mathrm{d}x + \frac{1}{16}\underbrace{\int_0^1 \frac{\ln^4(1-x^2)}{x}\mathrm{d}x}_{1-x^2=y}\end{aligned}$$

$$-\frac{5}{16}\int_0^1 \frac{\ln^4\left(\frac{1-x}{1+x}\right)}{x}\mathrm{d}x - \underbrace{\frac{1}{2}\int_0^1 \frac{\ln^3\left(\frac{1-x}{1+x}\right)\ln(1+x)}{x}\mathrm{d}x}_{(1-x)/(1+x)=y}$$

$$= \frac{9}{32}\int_0^1 \frac{\ln^4(y)}{1-y}\mathrm{d}y - \frac{1}{4}\int_0^1 \frac{\ln^4(1+x)}{x}\mathrm{d}x - \frac{5}{16}\int_0^1 \frac{\ln^4\left(\frac{1-x}{1+x}\right)}{x}\mathrm{d}x$$
$$+\int_0^1 \frac{\ln^3(y)\ln(1+x)}{1-y^2}\mathrm{d}y - \ln(2)\int_0^1 \frac{\ln^3(y)}{1-y^2}\mathrm{d}y.$$

These integrals are given in (3.8), (3.20), (3.15), (3.51), and (3.14) respectively.

3.3.31 $\int_0^1 \frac{\ln(1-x)\ln^3(1+x)}{x}\mathrm{d}x$

Show that

$$\int_0^1 \frac{\ln(1-x)\ln^3(1+x)}{x}\mathrm{d}x = -6\operatorname{Li}_5\left(\frac{1}{2}\right) - 6\ln(2)\operatorname{Li}_4\left(\frac{1}{2}\right) + \frac{3}{4}\zeta(5)$$
$$+\frac{21}{8}\zeta(2)\zeta(3) - \frac{21}{8}\ln^2(2)\zeta(3) + \ln^3(2)\zeta(2) - \frac{1}{5}\ln^5(2). \qquad (3.124)$$

Solution. Make use of

$$\ln(1-x)\ln^3(1+x)$$
$$= \frac{1}{8}\ln^4(1-x^2) - \frac{1}{8}\ln^4\left(\frac{1-x}{1+x}\right) - \ln^3(1-x)\ln(1+x),$$

we get

$$\int_0^1 \frac{\ln(1-x)\ln^3(1+x)}{x}\mathrm{d}x$$
$$= \underbrace{\frac{1}{8}\int_0^1 \frac{\ln^4(1-x^2)}{x}\mathrm{d}x}_{1-x^2=y} - \frac{1}{8}\int_0^1 \frac{\ln^4\left(\frac{1-x}{1+x}\right)}{x}\mathrm{d}x - \int_0^1 \frac{\ln^3(1-x)\ln(1+x)}{x}\mathrm{d}x$$
$$= \frac{1}{16}\int_0^1 \frac{\ln^4(y)}{1-y}\mathrm{d}y - \frac{1}{8}\int_0^1 \frac{\ln^4\left(\frac{1-x}{1+x}\right)}{x}\mathrm{d}x - \int_0^1 \frac{\ln^3(1-x)\ln(1+x)}{x}\mathrm{d}x.$$

These integrals are calculated in (3.8), (3.15), and (3.123).

3.3.32 $\int_0^1 \frac{\ln^3(1+x)\ln(x)}{x}\mathrm{d}x$

Show that

$$\int_0^1 \frac{\ln^3(1+x)\ln(x)}{x}\mathrm{d}x = -12\operatorname{Li}_5\left(\frac{1}{2}\right) - 12\ln(2)\operatorname{Li}_4\left(\frac{1}{2}\right) + \frac{99}{16}\zeta(5)$$
$$+3\zeta(2)\zeta(3) - \frac{21}{4}\ln^2(2)\zeta(3) + 2\ln^3(2)\zeta(2) - \frac{2}{5}\ln^5(2). \qquad (3.125)$$

Solution. Put $x = \frac{1-t}{t}$,

$$\int_0^1 \frac{\ln^3(1+x)\ln(x)}{x}\mathrm{d}x = -\int_{\frac{1}{2}}^1 \frac{\ln^3(t)\ln(\frac{1-t}{t})}{t(1-t)}\mathrm{d}t$$
$$= \int_{\frac{1}{2}}^1 \frac{\ln^4(t)}{t}\mathrm{d}t + \int_{\frac{1}{2}}^1 \frac{\ln^4(t)}{1-t}\mathrm{d}t - \underbrace{\int_{\frac{1}{2}}^1 \frac{\ln^3(t)\ln(1-t)}{t}\mathrm{d}t}_{\text{IBP}}$$
$$-\int_{\frac{1}{2}}^1 \frac{\ln^3(t)\ln(1-t)}{1-t}\mathrm{d}t$$
$$= -\frac{\ln^5(2)}{20} + \frac{3}{4}\underbrace{\int_{\frac{1}{2}}^1 \frac{\ln^4(t)}{1-t}\mathrm{d}t}_{\int_0^1 - \int_0^{1/2}} - \underbrace{\int_{\frac{1}{2}}^1 \frac{\ln^3(t)\ln(1-t)}{1-t}\mathrm{d}t}_{\int_0^1 - \int_0^{1/2}}$$
$$= -\frac{\ln^5(2)}{20} + \frac{3}{4}\int_0^1 \frac{\ln^4(t)}{1-t}\mathrm{d}t - \frac{3}{4}\int_0^{\frac{1}{2}} \frac{\ln^4(t)}{1-t}\mathrm{d}t$$
$$-\int_0^1 \frac{\ln^3(t)\ln(1-t)}{1-t}\mathrm{d}t + \int_0^{\frac{1}{2}} \frac{\ln^3(t)\ln(1-t)}{1-t}\mathrm{d}t$$
$$\{\text{recall the relation involving the last integral from (3.120)}\}$$
$$= \frac{\ln^5(2)}{10} - \frac{93}{16}\zeta(5) + \frac{3}{4}\int_0^1 \frac{\ln^4(t)}{1-t}\mathrm{d}t - \frac{1}{2}\int_0^{\frac{1}{2}} \frac{\ln^4(t)}{1-t}\mathrm{d}t$$
$$-\frac{1}{2}\int_0^1 \frac{\ln^3(t)\ln(1-t)}{1-t}\mathrm{d}t.$$

These integrals are computed in (3.8), (3.19), and (3.41) respectively.

A different approach involving the derivative of the beta function may be found in [30].

3.3.33 $\int_0^1 \frac{\mathrm{Li}_2(-x)}{1+x^2}\mathrm{d}x$

Show that

$$\int_0^1 \frac{\mathrm{Li}_2(-x)}{1+x^2}\mathrm{d}x = \frac{7\pi^3}{96} + \frac{3\pi}{16}\ln^2(2) + \frac{3}{2}\ln(2)G - 3\,\Im\,\mathrm{Li}_3(1+i). \quad (3.126)$$

Solution. Write $\mathrm{Li}_2(-x) = \int_0^1 \frac{x\ln(y)}{1+xy}\mathrm{d}y$, which follows from (1.75), we have

$$\begin{aligned}
\int_0^1 \frac{\mathrm{Li}_2(-x)}{1+x^2}\mathrm{d}x &= \int_0^1 \frac{1}{1+x^2}\left(\int_0^1 \frac{x\ln(y)}{1+xy}\mathrm{d}y\right)\mathrm{d}x \\
&\quad \{\text{change the order of integration}\} \\
&= \int_0^1 \ln(y)\left(\int_0^1 \frac{x}{(1+x^2)(1+xy)}\mathrm{d}x\right)\mathrm{d}y \\
&\quad \{\text{evaluate the inner integral by partial fraction decomposition}\} \\
&= \int_0^1 \ln(y)\left(\frac{\pi}{4}\frac{y}{1+y^2} + \frac{\ln(2)}{2}\frac{1}{1+y^2} - \frac{\ln(1+y)}{1+y^2}\right)\mathrm{d}y \\
&= \frac{\pi}{4}\underbrace{\int_0^1 \frac{y\ln(y)}{1+y^2}\mathrm{d}y}_{y=\sqrt{x}} + \frac{\ln(2)}{2}\int_0^1 \frac{\ln(y)}{1+y^2}\mathrm{d}y - \int_0^1 \frac{\ln(y)\ln(1+y)}{1+y^2}\mathrm{d}y \\
&= \frac{\pi}{16}\int_0^1 \frac{\ln(x)}{1+x}\mathrm{d}x + \frac{\ln(2)}{2}\int_0^1 \frac{\ln(y)}{1+y^2}\mathrm{d}y - \int_0^1 \frac{\ln(y)\ln(1+y)}{1+y^2}\mathrm{d}y. \quad (3.127)
\end{aligned}$$

The solution finalizes on recalling the results (3.11), (1.163), and (3.109).

3.3.34 $\int_0^1 \frac{\mathrm{Li}_2^2(-x)}{x}\mathrm{d}x$

Show that

$$\int_0^1 \frac{\mathrm{Li}_2^2(-x)}{x}\mathrm{d}x = \frac{3}{4}\zeta(2)\zeta(3) - \frac{17}{16}\zeta(5). \quad (3.128)$$

Solution. By using (1.78):

$$\mathrm{Li}_2(-x) = \frac{1}{2}\mathrm{Li}_2(x^2) - \mathrm{Li}_2(x),$$

we write

$$\int_0^1 \frac{\operatorname{Li}_2^2(-x)}{x}\mathrm{d}x = \int_0^1 \frac{1}{x}\left(\frac{1}{2}\operatorname{Li}_2(x^2)-\operatorname{Li}_2(x)\right)^2 \mathrm{d}x$$

$$= \frac{1}{4}\underbrace{\int_0^1 \frac{\operatorname{Li}_2^2(x^2)}{x}\mathrm{d}x}_{x^2\to x} - \int_0^1 \frac{\operatorname{Li}_2(x^2)\operatorname{Li}_2(x)}{x}\mathrm{d}x + \int_0^1 \frac{\operatorname{Li}_2^2(x)}{x}\mathrm{d}x$$

$$= \frac{1}{8}\int_0^1 \frac{\operatorname{Li}_2^2(x)}{x}\mathrm{d}x - \int_0^1 \frac{\operatorname{Li}_2(x^2)\operatorname{Li}_2(x)}{x}\mathrm{d}x + \int_0^1 \frac{\operatorname{Li}_2^2(x)}{x}\mathrm{d}x$$

$$= \frac{9}{8}\int_0^1 \frac{\operatorname{Li}_2^2(x)}{x}\mathrm{d}x - \int_0^1 \frac{\operatorname{Li}_2(x^2)\operatorname{Li}_2(x)}{x}\mathrm{d}x$$

$$\{\text{expand } \operatorname{Li}_2(x) \text{ and } \operatorname{Li}_2(x^2) \text{ in series}\}$$

$$= \frac{9}{8}\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^1 x^{n-1}\operatorname{Li}_2(x)\mathrm{d}x - \sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^1 x^{2n-1}\operatorname{Li}_2(x)\mathrm{d}x$$

$$\{\text{recall the result (3.66) for both integrals}\}$$

$$= \frac{9}{8}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\frac{\zeta(2)}{n}-\frac{H_n}{n^2}\right) - \sum_{n=1}^{\infty}\frac{1}{n^2}\left(\frac{\zeta(2)}{2n}-\frac{H_{2n}}{(2n)^2}\right)$$

$$= \frac{9}{8}\zeta(2)\zeta(3) - \frac{9}{8}\sum_{n=1}^{\infty}\frac{H_n}{n^4} - \frac{1}{2}\zeta(2)\zeta(3) + 4\sum_{n=1}^{\infty}\frac{H_{2n}}{(2n)^4}$$

$$\{\text{make use of (1.5) for the latter sum}\}$$

$$= \frac{5}{8}\zeta(2)\zeta(3) - \frac{9}{8}\sum_{n=1}^{\infty}\frac{H_n}{n^4} + 4\left(\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n}{n^4}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^4}\right)$$

$$= \frac{5}{8}\zeta(2)\zeta(3) + \frac{7}{8}\sum_{n=1}^{\infty}\frac{H_n}{n^4} + 2\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^4}. \tag{3.129}$$

Gather the results (4.4) and (4.93) to finish the solution.

3.3.35 $\int_0^{\frac{1}{2}} \frac{\operatorname{Li}_2^2(x)}{x}\mathrm{d}x$

The following integral is proposed by Cornel Vălean (see[38]):

$$\int_0^{\frac{1}{2}} \frac{\operatorname{Li}_2^2(x)}{x}\mathrm{d}x = -2\operatorname{Li}_5\left(\frac{1}{2}\right) - 2\ln(2)\operatorname{Li}_4\left(\frac{1}{2}\right) + \frac{27}{32}\zeta(5) - \frac{5}{8}\ln(2)\zeta(4)$$
$$+\frac{7}{8}\zeta(2)\zeta(3) - \frac{7}{8}\ln^2(2)\zeta(3) + \frac{1}{2}\ln^3(2)\zeta(2) - \frac{7}{60}\ln^5(2). \tag{3.130}$$

Solution. By applying integration by parts twice, we have

$$\int_0^{\frac{1}{2}} \frac{\operatorname{Li}_2^2(x)}{x}\mathrm{d}x = \operatorname{Li}_3(x)\operatorname{Li}_2(x)\Big|_0^{\frac{1}{2}} + \int_0^{\frac{1}{2}} \frac{\operatorname{Li}_3(x)\ln(1-x)}{x}\mathrm{d}x$$

$$= \operatorname{Li}_3\left(\frac{1}{2}\right)\operatorname{Li}_2\left(\frac{1}{2}\right) + \operatorname{Li}_4(x)\ln(1-x)\Big|_0^{\frac{1}{2}} + \int_0^{\frac{1}{2}} \frac{\operatorname{Li}_4(x)}{1-x}\mathrm{d}x$$

$$\left\{\text{use } \frac{\operatorname{Li}_4(x)}{1-x} = \sum_{n=1}^{\infty} H_{n-1}^{(4)} x^{n-1} \text{ given in (2.3)}\right\}$$

$$= \operatorname{Li}_3\left(\frac{1}{2}\right)\operatorname{Li}_2\left(\frac{1}{2}\right) - \ln(2)\operatorname{Li}_4\left(\frac{1}{2}\right) + \sum_{n=1}^{\infty} H_{n-1}^{(4)} \int_0^{\frac{1}{2}} x^{n-1}\mathrm{d}x$$

$$\left\{\text{write } \int_0^{\frac{1}{2}} x^{n-1}\mathrm{d}x = \frac{1}{n2^n} \text{ and } H_{n-1}^{(4)} = H_n^{(4)} - \frac{1}{n^4}\right\}$$

$$= \operatorname{Li}_3\left(\frac{1}{2}\right)\operatorname{Li}_2\left(\frac{1}{2}\right) - \ln(2)\operatorname{Li}_4\left(\frac{1}{2}\right) + \sum_{n=1}^{\infty} \frac{H_n^{(4)} - \frac{1}{n^4}}{n2^n}$$

$$= \operatorname{Li}_3\left(\frac{1}{2}\right)\operatorname{Li}_2\left(\frac{1}{2}\right) - \ln(2)\operatorname{Li}_4\left(\frac{1}{2}\right) - \operatorname{Li}_5\left(\frac{1}{2}\right) + \sum_{n=1}^{\infty} \frac{H_n^{(4)}}{n2^n}$$

$$\{\text{set } q = 3 \text{ in (3.87) to get the integral representation of the sum}\}$$

$$= \operatorname{Li}_3\left(\frac{1}{2}\right)\operatorname{Li}_2\left(\frac{1}{2}\right) - \ln(2)\operatorname{Li}_4\left(\frac{1}{2}\right) - \operatorname{Li}_5\left(\frac{1}{2}\right)$$
$$-\frac{1}{6}\int_0^1 \frac{\ln^3(1-x)\ln(1+x)}{x}\mathrm{d}x,$$

This integral is given in (3.123) and the values of $\operatorname{Li}_2\left(\frac{1}{2}\right)$ and $\operatorname{Li}_3\left(\frac{1}{2}\right)$ are given in (1.81) and (1.92) respectively.

3.3.36 $\int_0^1 \frac{\ln^3(1-x)\operatorname{Li}_2(x)}{x}\mathrm{d}x$

Show that

$$\int_0^1 \frac{\ln^3(1-x)\operatorname{Li}_2(x)}{x}\mathrm{d}x = -\frac{1}{2}\zeta(6) - 6\zeta^2(3). \tag{3.131}$$

Solution. With subbing $1 - x = y$, we have

$$\int_0^1 \frac{\ln^3(1-x)\operatorname{Li}_2(x)}{x}\mathrm{d}x = \int_0^1 \frac{\ln^3(y)\operatorname{Li}_2(1-y)}{1-y}\mathrm{d}y$$

$$\{\text{recall the dilogarithm reflection formula (1.80)}\}$$

$$= \int_0^1 \frac{\ln^3(y)}{1-y}(\zeta(2) - \ln(y)\ln(1-y) - \operatorname{Li}_2(y))\mathrm{d}y$$

$$= \zeta(2)\int_0^1 \frac{\ln^3(y)}{1-y}\mathrm{d}y - \int_0^1 \frac{\ln^4(y)\ln(1-y)}{1-y}\mathrm{d}y - \int_0^1 \frac{\ln^3(y)\operatorname{Li}_2(y)}{1-y}\mathrm{d}y$$

$$\left\{\text{expand } \frac{\operatorname{Li}_2(y)}{1-y} \text{ in series as shown in (2.3)}\right\}$$

$$= \zeta(2)\int_0^1 \frac{\ln^3(y)}{1-y}\mathrm{d}y - \int_0^1 \frac{\ln^4(y)\ln(1-y)}{1-y}\mathrm{d}y - \sum_{n=1}^{\infty} H_{n-1}^{(2)} \int_0^1 y^{n-1}\ln^3(y)\mathrm{d}y$$

$$= \zeta(2)\int_0^1 \frac{\ln^3(y)}{1-y}\mathrm{d}y - \int_0^1 \frac{\ln^4(y)\ln(1-y)}{1-y}\mathrm{d}y - \sum_{n=1}^{\infty}\left(H_n^{(2)} - \frac{1}{n^2}\right)\left(\frac{-3!}{n^4}\right)$$

$$= \zeta(2)\int_0^1 \frac{\ln^3(y)}{1-y}\mathrm{d}y - \int_0^1 \frac{\ln^4(y)\ln(1-y)}{1-y}\mathrm{d}y + 6\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4} - 6\zeta(6).$$

These terms are given in (3.8), (3.41), and (4.56) respectively.

Chapter 4

Harmonic Series

4.1 Generalized Harmonic Series

4.1.1 $\sum_{n=1}^{\infty} \frac{H_{\frac{n}{p}}}{n^q}$

Let $p \in \mathbb{Z}^+$ and $q \geq 2$. Then the following identity holds:

$$\sum_{n=1}^{\infty} \frac{H_{\frac{n}{p}}}{n^q} = (-1)^q p \sum_{n=1}^{\infty} \frac{H_{pn}}{(pn)^q} - \sum_{k=1}^{q-2} (-p)^{-k} \zeta(q-k)\zeta(k+1). \tag{4.1}$$

Proof.

$$\sum_{n=1}^{\infty} \frac{H_{\frac{n}{p}}}{n^q} = \frac{1}{p} \sum_{n=1}^{\infty} \frac{1}{n^{q-1}} \left(\frac{H_{\frac{n}{p}}}{\frac{n}{p}} \right)$$

$$\{\text{replace } n \text{ with } n/p \text{ in the identity (1.124)}\}$$

$$= \frac{1}{p} \sum_{n=1}^{\infty} \frac{1}{n^{q-1}} \left(-\int_0^1 x^{\frac{n}{p}-1} \ln(1-x) \mathrm{d}x \right)$$

$$= -\frac{1}{p} \int_0^1 \frac{\ln(1-x)}{x} \left(\sum_{n=1}^{\infty} \frac{x^{\frac{n}{p}}}{n^{q-1}} \right) \mathrm{d}x$$

$$= -\frac{1}{p} \int_0^1 \frac{\ln(1-x)}{x} \left(\operatorname{Li}_{q-1}(x^{\frac{1}{p}}) \right) \mathrm{d}x$$

$$\overset{x=y^p}{=} -\int_0^1 \frac{\ln(1-y^p) \operatorname{Li}_{q-1}(y)}{y} \mathrm{d}y$$

$$\{\text{expand } \ln(1-y^p) \text{ in series}\}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 y^{pn-1} \operatorname{Li}_{q-1}(y) \mathrm{d}y$$

$$\{\text{make use of (3.65) for the integral}\}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \left((-1)^q \frac{H_{pn}}{(pn)^{q-1}} - \sum_{k=1}^{q-2} (-1)^k \frac{\zeta(q-k)}{(pn)^k} \right)$$

$$= (-1)^q p \sum_{n=1}^{\infty} \frac{H_{pn}}{(pn)^q} - \sum_{k=1}^{q-2} (-p)^{-k} \zeta(q-k) \left(\sum_{n=1}^{\infty} \frac{1}{n^{k+1}} \right)$$

$$= (-1)^q p \sum_{n=1}^{\infty} \frac{H_{pn}}{(pn)^q} - \sum_{k=1}^{q-2} (-p)^{-k} \zeta(q-k) \zeta(k+1)$$

and the proof is finalized.

4.1.2 $\sum_{n=1}^{\infty} \frac{H_n}{(n+s)^q}$

Let $s \notin \mathbb{Z}^-$ and $q \in \mathbb{Z}_{>1}$. Then the following identity holds:

$$\sum_{n=1}^{\infty} \frac{H_n}{(n+s)^q} = \frac{(-1)^q}{(q-1)!} \left(\psi^{(q-1)}(s)(\psi(s)+\gamma) - \frac{1}{2}\psi^{(q)}(s) \right) + \frac{(-1)^q}{(q-1)!} \sum_{k=1}^{q-2} \binom{q-2}{k} \psi^{(q-k-1)}(s) \psi^{(k)}(s). \tag{4.2}$$

Proof.

$$\sum_{n=1}^{\infty} \frac{H_n}{(n+s)^q} = \sum_{n=1}^{\infty} H_n \left(\frac{(-1)^{q-1}}{(q-1)!} \int_0^1 x^{n+s-1} \ln^{q-1}(x) \mathrm{d}x \right)$$

$$= \frac{(-1)^{q-1}}{(q-1)!} \int_0^1 x^{s-1} \ln^{q-1}(x) \left(\sum_{n=1}^{\infty} H_n x^n \right) \mathrm{d}x$$

$$\{\text{recall the generating function (2.4)}\}$$

$$= \frac{(-1)^{q-1}}{(q-1)!} \int_0^1 x^{s-1} \ln^{q-1}(x) \left(-\frac{\ln(1-x)}{1-x} \right) \mathrm{d}x$$

$$= \frac{(-1)^q}{(q-1)!} \int_0^1 \frac{x^{s-1} \ln^{q-1}(x) \ln(1-x)}{1-x} \mathrm{d}x. \tag{4.3}$$

This integral is given in (3.40).

4.1.3 $\sum_{n=1}^{\infty}\frac{H_n}{n^q}$

Let $q \in \mathbb{Z}_{>1}$. Then the following identity holds:

$$\sum_{n=1}^{\infty}\frac{H_n}{n^q} = \frac{1}{2}(q+2)\zeta(q+1) - \frac{1}{2}\sum_{k=1}^{q-2}\zeta(q-k)\zeta(k+1). \tag{4.4}$$

Proof (i). Set $s = 1$ in (4.3),

$$\begin{aligned}\frac{(-1)^q}{(q-1)!}\int_0^1 \frac{\ln^{q-1}(x)\ln(1-x)}{1-x}\mathrm{d}x &= \sum_{n=1}^{\infty}\frac{H_n}{(n+1)^q} = \sum_{n=1}^{\infty}\frac{H_{n-1}}{n^q}\\ &= \sum_{n=1}^{\infty}\frac{H_n - \frac{1}{n}}{n^q} = \sum_{n=1}^{\infty}\frac{H_n}{n^q} - \zeta(q+1).\end{aligned}$$

The integral on the LHS is given in (3.41).

The following second proof is by Rob Johnson and may be found in [14]:

Proof (ii). Start with expanding $\zeta(q-k)$ and $\zeta(k+1)$ in series,

$$\zeta(q-k)\zeta(k+1) = \left(\sum_{m=1}^{\infty}\frac{1}{m^{q-k}}\right)\left(\sum_{n=1}^{\infty}\frac{1}{n^{k+1}}\right).$$

Next, take the summation for both sides from $k = 1$ to $q - 2$,

$$\begin{aligned}\sum_{k=1}^{q-2}\zeta(q-k)\zeta(k+1) &= \sum_{k=1}^{q-2}\left(\sum_{m=1}^{\infty}\frac{1}{m^{q-k}}\right)\left(\sum_{n=1}^{\infty}\frac{1}{n^{k+1}}\right)\\ &\qquad \{\text{change the order of summations}\}\\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{q-2}\frac{1}{m^{q-k}n^{k+1}}\\ &\qquad \{\text{break up the middle sum}\}\\ &= \sum_{m=1}^{\infty}\left(\sum_{n=1}^{m-1} + a_{n=m} + \sum_{n=m+1}^{\infty}\right)\sum_{k=1}^{q-2}\frac{1}{m^{q-k}n^{k+1}}\\ &\qquad \{\text{pull out the terms for } n=m\}\\ &= \sum_{m=1}^{\infty}\sum_{k=1}^{q-2}\frac{1}{m^{q+1}} + \sum_{m=1}^{\infty}\left(\sum_{n=1}^{m-1} + \sum_{n=m+1}^{\infty}\right)\sum_{k=1}^{q-2}\frac{1}{m^{q-k}n^{k+1}}\\ &= \sum_{k=1}^{q-2}\left(\sum_{m=1}^{\infty}\frac{1}{m^{q+1}}\right) + \sum_{m=1}^{\infty}\left(\sum_{n=1}^{m-1} + \sum_{n=m+1}^{\infty}\right)\frac{1}{m^q n}\left(\sum_{k=1}^{q-2}\frac{m^k}{n^k}\right)\end{aligned}$$

$$=\sum_{k=1}^{q-2}\zeta(q+1)+\sum_{m=1}^{\infty}\left(\sum_{n=1}^{m-1}+\sum_{n=m+1}^{\infty}\right)\left(\frac{1}{nm^{q-1}(n-m)}-\frac{1}{mn^{q-1}(n-m)}\right)$$

$$=(q-2)\zeta(q+1)+\sum_{m=1}^{\infty}\sum_{n=1}^{m-1}\left(\frac{1}{nm^{q-1}(n-m)}-\frac{1}{mn^{q-1}(n-m)}\right)$$

$$+\sum_{m=1}^{\infty}\sum_{n=m+1}^{\infty}\left(\frac{1}{nm^{q-1}(n-m)}-\frac{1}{mn^{q-1}(n-m)}\right).$$

By using $\sum_{m=1}^{\infty}\sum_{n=1}^{m-1}a_mb_n=\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty}a_mb_n$ given in (1.17), the first double sum becomes

$$\sum_{m=1}^{\infty}\sum_{n=1}^{m-1}\left(\frac{1}{nm^{q-1}(n-m)}-\frac{1}{mn^{q-1}(n-m)}\right)$$

$$=\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty}\left(\frac{1}{nm^{q-1}(n-m)}-\frac{1}{mn^{q-1}(n-m)}\right)$$

$$\{\text{swab the variables } n \text{ and } m\}$$

$$=\sum_{m=1}^{\infty}\sum_{n=m+1}^{\infty}\left(\frac{1}{nm^{q-1}(n-m)}-\frac{1}{mn^{q-1}(n-m)}\right).$$

Therefore, our sum simplifies to

$$\sum_{k=1}^{q-2}\zeta(q-k)\zeta(k+1)=(q-2)\zeta(q+1)$$

$$+2\sum_{m=1}^{\infty}\sum_{n=m+1}^{\infty}\left(\frac{1}{nm^{q-1}(n-m)}-\frac{1}{mn^{q-1}(n-m)}\right)$$

$$\{\text{shift the index } n \text{ by } +m\}$$

$$=(q-2)\zeta(q+1)+2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{1}{(n+m)m^{q-1}n}-\frac{1}{m(n+m)^{q-1}n}\right)$$

$$=(q-2)\zeta(q+1)+2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(n+m)m^{q-1}n}-2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m(n+m)^{q-1}n}.$$

First sum:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(n+m)m^{q-1}n}=\sum_{m=1}^{\infty}\frac{1}{m^q}\left(\sum_{n=1}^{\infty}\frac{m}{n(n+m)}\right)$$

$$\{\text{recall the definition of } H_m \text{ (1.129)}\}$$

$$= \sum_{m=1}^{\infty} \frac{H_m}{m^q}.$$

Second sum: Multiply the summand by $\frac{n+m}{n+m}$,

$$\begin{aligned}
\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{m(n+m)^{q-1}n} &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{n+m}{m(n+m)^q n} \\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{m(n+m)^q} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{(n+m)^q n}
\end{aligned}$$

$$\{\text{swap the variables } m \text{ and } n \text{ in the first double sum}\}$$

$$\{\text{and change the order of summations in the second double sum}\}$$

$$= 2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{n(n+m)^q}$$

$$\{\text{shift the index } m \text{ by } -n\}$$

$$= 2\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty} \frac{1}{nm^q}$$

$$\left\{\text{use} \sum_{m=n+1}^{\infty} a_m = \sum_{m=n}^{\infty} a_m - a_n \text{ for the inner sum}\right\}$$

$$= 2\sum_{n=1}^{\infty}\left(\sum_{m=n}^{\infty} \frac{1}{nm^q} - \frac{1}{n^{q+1}}\right)$$

$$= 2\sum_{n=1}^{\infty}\sum_{m=n}^{\infty} \frac{1}{nm^q} - 2\sum_{n=1}^{\infty} \frac{1}{n^{q+1}}$$

$$\left\{\text{use} \sum_{n=1}^{\infty}\sum_{m=n}^{\infty} a_m b_n = \sum_{m=1}^{\infty}\sum_{n=1}^{m} a_m b_n \text{ given in (1.16)}\right\}$$

$$= 2\sum_{m=1}^{\infty}\sum_{n=1}^{m} \frac{1}{nm^q} - 2\zeta(q+1)$$

$$= 2\sum_{m=1}^{\infty} \frac{1}{m^q}\left(\sum_{n=1}^{m} \frac{1}{n}\right) - 2\zeta(q+1)$$

$$= 2\sum_{m=1}^{\infty} \frac{H_m}{m^q} - 2\zeta(q+1).$$

Collect the two sums to finish the proof.

Let's replace q with $2q$ in (4.4),

$$\sum_{n=1}^{\infty}\frac{H_n}{n^{2q}}=(q+1)\zeta(2q+1)-\frac{1}{2}\sum_{k=1}^{2q-2}\zeta(2q-k)\zeta(k+1)$$

$$\{\text{this sum is simplified in (3.42)}\}$$

$$=(q+1)\zeta(2q+1)-\sum_{k=1}^{q-1}\zeta(2q-2k+1)\zeta(2k). \tag{4.5}$$

Examples:

$$\sum_{n=1}^{\infty}\frac{H_n}{n^2}=2\zeta(3);$$

$$\sum_{n=1}^{\infty}\frac{H_n}{n^4}=3\zeta(5)-\zeta(2)\zeta(3);$$

$$\sum_{n=1}^{\infty}\frac{H_n}{n^6}=4\zeta(7)-\zeta(2)\zeta(5)-\zeta(3)\zeta(4);$$

$$\sum_{n=1}^{\infty}\frac{H_n}{n^8}=5\zeta(9)-\zeta(2)\zeta(7)-\zeta(3)\zeta(6)-\zeta(4)\zeta(5).$$

A special case of Euler sum is

$$\sum_{n=1}^{\infty}\frac{H_n}{n^{2q+1}}=-\frac{1}{2}\sum_{k=1}^{2q-1}(-1)^k\zeta(2q-k+1)\zeta(k+1). \tag{4.6}$$

To show that, replace q with $=2q$ in (3.65),

$$\frac{H_n}{n^{2q}}=-\int_0^1 x^{n-1}\operatorname{Li}_{2q}(x)\mathrm{d}x-\sum_{k=1}^{2q-1}\frac{(-1)^k}{n^k}\zeta(2q-k+1).$$

Divide both sides by n then consider the summation over $n\ge 1$,

$$\sum_{n=1}^{\infty}\frac{H_n}{n^{2q+1}}$$

$$=-\int_0^1\frac{\operatorname{Li}_{2q}(x)}{x}\left(\sum_{n=1}^{\infty}\frac{x^n}{n}\right)\mathrm{d}x-\sum_{k=1}^{2q-1}(-1)^k\zeta(2q-k+1)\sum_{n=1}^{\infty}\frac{1}{n^{k+1}}$$

$$=\int_0^1\frac{\operatorname{Li}_{2q}(x)\ln(1-x)}{x}\mathrm{d}x-\sum_{k=1}^{2q-1}(-1)^k\zeta(2q-k+1)\zeta(k+1)$$

$$\{\text{expand } \operatorname{Li}_{2q}(x) \text{ in series}\}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{2q}} \int_0^1 x^{n-1} \ln(1-x) \mathrm{d}x - \sum_{k=1}^{2q-1} (-1)^k \zeta(2q-k+1)\zeta(k+1)$$

$$\{\text{make use of the result } (1.124)\}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{2q}} \left(-\frac{H_n}{n}\right) - \sum_{k=1}^{2q-1} (-1)^k \zeta(2q-k+1)\zeta(k+1)$$

$$\{\text{add the sum to both sides then divide by } 2\}$$

$$= -\frac{1}{2} \sum_{k=1}^{2q-1} (-1)^k \zeta(2q-k+1)\zeta(k+1).$$

Examples:

$$\sum_{n=1}^{\infty} \frac{H_n}{n^3} = \frac{5}{4}\zeta(4);$$

$$\sum_{n=1}^{\infty} \frac{H_n}{n^5} = \frac{7}{4}\zeta(6) - \frac{1}{2}\zeta^2(3);$$

$$\sum_{n=1}^{\infty} \frac{H_n}{n^7} = \frac{9}{4}\zeta(8) - \zeta(3)\zeta(5);$$

$$\sum_{n=1}^{\infty} \frac{H_n}{n^9} = \frac{11}{4}\zeta(10) - \zeta(3)\zeta(7) - \frac{1}{2}\zeta^2(5).$$

4.1.4 $\sum_{n=1}^{\infty} \frac{\overline{H}_n}{n^q}$

Let $q \in \mathbb{Z}_{>1}$. Then the following identity holds:

$$\sum_{n=1}^{\infty} \frac{\overline{H}_n}{n^q} = 2\left(1 - \frac{1}{2^q}\right) \ln(2)\zeta(q) + \left(1 - \frac{1}{2^q} - \frac{q}{2}\right) \zeta(q+1)$$
$$+ \frac{1}{2} \sum_{k=1}^{q-2} \eta(q-k)\eta(k+1). \tag{4.7}$$

Proof (i).

$$\sum_{n=1}^{\infty} \frac{\overline{H}_n}{n^q} = \sum_{n=1}^{\infty} \overline{H}_n \left(\frac{(-1)^{q-1}}{(q-1)!} \int_0^1 x^{n-1} \ln^{q-1}(x) \mathrm{d}x \right)$$

$$= \frac{(-1)^{q-1}}{(q-1)!} \int_0^1 \frac{\ln^{q-1}(x)}{x} \left(\sum_{n=1}^{\infty} \overline{H}_n x^n \right) \mathrm{d}x$$

$$\{\text{recall the generation function (2.29)}\}$$

$$= \frac{(-1)^{q-1}}{(q-1)!} \int_0^1 \frac{\ln^{q-1}(x)}{x} \left(\frac{\ln(1+x)}{1-x} \right) \mathrm{d}x$$

$$= \frac{(-1)^{q-1}}{(q-1)!} \int_0^1 \frac{\ln^{q-1}(x)\ln(1+x)}{x(1-x)} \mathrm{d}x \tag{4.8}$$

and this integral is found in (3.48).

Proof (ii). We apply Rob's approach, but here we begin with expanding $\eta(q-k)$ and $\eta(k+1)$ in series,

$$\sum_{k=1}^{q-2} \eta(q-k)\eta(k+1) = \sum_{k=1}^{q-2} \left(\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m^{q-k}} \right) \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{k+1}} \right)$$

$$\{\text{change the order of summations}\}$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{q-2} \frac{(-1)^{m+n}}{m^{q-k} n^{k+1}}$$

$$\{\text{break up the middle sum}\}$$

$$= \sum_{m=1}^{\infty} \left(\sum_{n=1}^{m-1} + a_{n=m} + \sum_{n=m+1}^{\infty} \right) \sum_{k=1}^{q-2} \frac{(-1)^{m+n}}{m^{q-k} n^{k+1}}$$

$$\{\text{pull out the terms for } n=m\}$$

$$= \sum_{m=1}^{\infty} \sum_{k=1}^{q-2} \frac{1}{m^{q+1}} + \sum_{m=1}^{\infty} \left(\sum_{n=1}^{m-1} + \sum_{n=m+1}^{\infty} \right) \sum_{k=1}^{q-2} \frac{(-1)^{m+n}}{m^{q-k} n^{k+1}}$$

$$= \sum_{k=1}^{q-2} \left(\sum_{m=1}^{\infty} \frac{1}{m^{q+1}} \right) + \sum_{m=1}^{\infty} \left(\sum_{n=1}^{m-1} + \sum_{n=m+1}^{\infty} \right) \frac{(-1)^{m+n}}{m^q n} \left(\sum_{k=1}^{q-2} \frac{m^k}{n^k} \right)$$

$$= \sum_{k=1}^{q-2} \zeta(q+1) + \sum_{m=1}^{\infty} \left(\sum_{n=1}^{m-1} + \sum_{n=m+1}^{\infty} \right) \left(\frac{(-1)^{m+n}}{nm^{q-1}(n-m)} - \frac{(-1)^{m+n}}{mn^{q-1}(n-m)} \right)$$

$$= (q-2)\zeta(q+1) + \sum_{m=1}^{\infty} \sum_{n=1}^{m-1} \left(\frac{(-1)^{m+n}}{nm^{q-1}(n-m)} - \frac{(-1)^{m+n}}{mn^{q-1}(n-m)} \right)$$

$$+ \sum_{m=1}^{\infty} \sum_{n=m+1}^{\infty} \left(\frac{(-1)^{m+n}}{nm^{q-1}(n-m)} - \frac{(-1)^{m+n}}{mn^{q-1}(n-m)} \right).$$

By using $\sum_{m=1}^{\infty}\sum_{n=1}^{m-1} a_m b_n = \sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty} a_m b_n$ given in (1.17), the first double sum becomes:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{m-1}\left(\frac{(-1)^{m+n}}{nm^{q-1}(n-m)}-\frac{(-1)^{m+n}}{mn^{q-1}(n-m)}\right)$$
$$=\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty}\left(\frac{(-1)^{m+n}}{nm^{q-1}(n-m)}-\frac{(-1)^{m+n}}{mn^{q-1}(n-m)}\right)$$
$$\{\text{swab the variables } n \text{ and } m\}$$
$$=\sum_{m=1}^{\infty}\sum_{n=m+1}^{\infty}\left(\frac{(-1)^{n+m}}{nm^{q-1}(n-m)}-\frac{(-1)^{n+m}}{mn^{q-1}(n-m)}\right).$$

Thus, our sum boils down to

$$\sum_{k=1}^{q-2}\eta(q-k)\eta(k+1)=(q-2)\zeta(q+1)$$
$$+2\sum_{m=1}^{\infty}\sum_{n=m+1}^{\infty}\left(\frac{(-1)^{n+m}}{nm^{q-1}(n-m)}-\frac{(-1)^{n+m}}{mn^{q-1}(n-m)}\right)$$
$$\{\text{shift the index } n \text{ by } +m\}$$
$$=(q-2)\zeta(q+1)+2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{(-1)^{n}}{(n+m)m^{q-1}n}-\frac{(-1)^{n}}{m(n+m)^{q-1}n}\right)$$
$$=(q-2)\zeta(q+1)+2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n}}{(n+m)m^{q-1}n}-2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n}}{m(n+m)^{q-1}n}.$$

First sum:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n}}{(n+m)m^{q-1}n}=\sum_{m=1}^{\infty}\frac{1}{m^q}\left(\sum_{n=1}^{\infty}\frac{(-1)^{n}m}{n(n+m)}\right)$$
$$=\sum_{m=1}^{\infty}\frac{1}{m^q}\left(\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n}-\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n+m}\right)$$
$$\{\text{recall the result of the second sum from (1.143)}\}$$
$$=\sum_{m=1}^{\infty}\frac{1}{m^q}\left(-\ln(2)-(-1)^m\left[\overline{H}_m-\ln(2)\right]\right)$$
$$=-\ln(2)\zeta(q)-\ln(2)\eta(q)-\sum_{m=1}^{\infty}\frac{(-1)^m\overline{H}_m}{m^q}.$$

Second sum: Multiply the summand by $\frac{n+m}{n+m}$,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^n}{m(n+m)^{q-1}n}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^n(n+m)}{m(n+m)^q n}$$

$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^n}{m(n+m)^q}+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+m)^q n}$$

$$\{\text{swap the variables } m \text{ and } n \text{ in the first double sum}\}$$

$$\{\text{and change the order of summations in the second double sum}\}$$

$$=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^m+(-1)^n}{n(n+m)^q}$$

$$\{\text{shift the index } m \text{ by } -n\}$$

$$=\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty}\frac{(-1)^{m-n}+(-1)^n}{nm^q}$$

$$\left\{\text{use }\sum_{m=n+1}^{\infty}a_m=\sum_{m=n}^{\infty}a_m-a_n\right\}$$

$$=\sum_{n=1}^{\infty}\left(\sum_{m=n}^{\infty}\frac{(-1)^{m-n}+(-1)^n}{nm^q}-\frac{1+(-1)^n}{n^{q+1}}\right)$$

$$=\sum_{n=1}^{\infty}\sum_{m=n}^{\infty}\frac{(-1)^{m-n}+(-1)^n}{nm^q}-\sum_{n=1}^{\infty}\frac{1+(-1)^n}{n^{q+1}}$$

$$\left\{\text{use }\sum_{n=1}^{\infty}\sum_{m=n}^{\infty}a_m b_n=\sum_{m=1}^{\infty}\sum_{n=1}^{m}a_m b_n \text{ given in (1.16)}\right\}$$

$$=\sum_{m=1}^{\infty}\sum_{n=1}^{m}\left(\frac{(-1)^{m-n}+(-1)^n}{nm^q}\right)-\zeta(q+1)+\eta(q+1)$$

$$=\sum_{m=1}^{\infty}\frac{1}{m^q}\left(\sum_{n=1}^{m}\frac{(-1)^{m-n}+(-1)^n}{n}\right)-\zeta(q+1)+\eta(q+1)$$

$$\{\text{recall the definition of } \overline{H}_n \text{ in (1.133) for the inner sum}\}$$

$$=\sum_{m=1}^{\infty}\frac{1}{m^q}\left(-(-1)^m\overline{H}_m-\overline{H}_m\right)-\zeta(q+1)+\eta(q+1)$$

$$=-\sum_{m=1}^{\infty}\frac{(-1)^m\overline{H}_m}{m^q}-\sum_{m=1}^{\infty}\frac{\overline{H}_m}{m^q}-\zeta(q+1)+\eta(q+1).$$

The proof finalizes on collecting the two sums then writing $\eta(s)=(1-2^{1-s})\zeta(s)$ given in (1.56).

Replace q with $2q$ in (4.7),

$$\sum_{n=1}^{\infty}\frac{\overline{H}_n}{n^{2q}} = 2\left(1-\frac{1}{2^{2q}}\right)\ln(2)\zeta(2q)+\left(1-\frac{1}{2^{2q}}-q\right)\zeta(2q+1)$$
$$+\frac{1}{2}\sum_{k=1}^{2q-2}\eta(2q-k)\eta(k+1)$$
$$\{\text{this sum is simplified in (3.49)}\}$$
$$= 2\left(1-\frac{1}{2^{2q}}\right)\ln(2)\zeta(2q)+\left(1-\frac{1}{2^{2q}}-q\right)\zeta(2q+1)$$
$$+\sum_{k=1}^{q-1}\eta(2q-2k+1)\eta(2k). \tag{4.9}$$

Examples:

$$\sum_{n=1}^{\infty}\frac{\overline{H}_n}{n^2} = \frac{3}{2}\ln(2)\zeta(2)-\frac{1}{4}\zeta(3);$$
$$\sum_{n=1}^{\infty}\frac{\overline{H}_n}{n^3} = \frac{7}{4}\ln(2)\zeta(3)-\frac{5}{16}\zeta(4);$$
$$\sum_{n=1}^{\infty}\frac{\overline{H}_n}{n^4} = \frac{15}{8}\ln(2)\zeta(4)+\frac{3}{8}\zeta(2)\zeta(3)-\frac{17}{16}\zeta(5);$$
$$\sum_{n=1}^{\infty}\frac{\overline{H}_n}{n^5} = \frac{31}{16}\ln(2)\zeta(5)+\frac{9}{32}\zeta^2(3)-\frac{49}{64}\zeta(6).$$

For other proofs, check [19, Theorem 3.5, p. 9] and [11, Theorem 7.1 (i), p. 32].

4.1.5 $\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^{2q}}$

Let $q\in\mathbb{Z}^+$. Then the following identity holds:

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^{2q}} = -\frac{1}{2}(2q+1)\eta(2q+1)+\sum_{k=0}^{q-1}\zeta(2q-2k+1)\eta(2k). \tag{4.10}$$

Proof (i).

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^{q}} = \sum_{n=1}^{\infty}(-1)^n H_n\left(\frac{(-1)^{q-1}}{(q-1)!}\int_0^1 x^{n-1}\ln^{q-1}(x)\mathrm{d}x\right)$$

$$= \frac{(-1)^{q-1}}{(q-1)!}\int_0^1 \frac{\ln^{q-1}(x)}{x}\left(\sum_{n=1}^{\infty} H_n(-x)^n\right)\mathrm{d}x$$

$$\{\text{replace } x \text{ with } -x \text{ in (2.4) to get this sum}\}$$

$$= \frac{(-1)^{q-1}}{(q-1)!}\int_0^1 \frac{\ln^{q-1}(x)}{x}\left(-\frac{\ln(1+x)}{1+x}\right)\mathrm{d}x$$

$$= \frac{(-1)^{q}}{(q-1)!}\int_0^1 \frac{\ln^{q-1}(x)\ln(1+x)}{x(1+x)}\mathrm{d}x. \tag{4.11}$$

Replace q with $2q$ in (4.11),

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^{2q}} = \frac{1}{(2q-1)!}\int_0^1 \frac{\ln^{2q-1}(x)\ln(1+x)}{x(1+x)}\mathrm{d}x.$$

This integral is given in (3.36).

Proof (ii). Using the integral form of $\overline{H}_n$ given in (1.140), we have

$$\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n}{n^{2q}} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^{2q}}\left(\ln(2) - \int_0^1 \frac{(-x)^n}{1+x}\mathrm{d}x\right)$$

$$= \ln(2)\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{2q}} - \sum_{n=1}^{\infty}\frac{1}{n^{2q}}\int_0^1\frac{x^n}{1+x}dx$$

$$\{\text{make use of the identity (3.1) for the integral}\}$$

$$= -\ln(2)\eta(2q) - \sum_{n=1}^{\infty}\frac{1}{n^{2q}}\left(\ln(2) + H_{\frac{n}{2}} - H_n\right)$$

$$= -\ln(2)\eta(2q) - \ln(2)\zeta(2q) - \sum_{n=1}^{\infty}\frac{H_{\frac{n}{2}}}{n^{2q}} + \sum_{n=1}^{\infty}\frac{H_n}{n^{2q}}.$$

The first sum can be simplified by setting $p = 2$ and replacing q with $2q$ in (4.1),

$$\sum_{n=1}^{\infty}\frac{H_{\frac{n}{2}}}{n^{2q}} = 2\sum_{n=1}^{\infty}\frac{H_{2n}}{(2n)^{2q}} - \sum_{k=1}^{2q-2}(-2)^{-k}\zeta(2q-k)\zeta(k+1)$$

$$\{\text{make use of (1.5) for the first sum}\}$$

$$= \sum_{n=1}^{\infty}\frac{H_n}{n^{2q}} + \sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^{2q}} - \sum_{k=1}^{2q-2}(-2)^{-k}\zeta(2q-k)\zeta(k+1). \tag{4.12}$$

Substituting this sum yields

$$\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n}{n^{2q}} + \sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^{2q}} = \sum_{k=1}^{2q-2}(-2)^{-k}\zeta(2q-k)\zeta(k+1)$$

$$-\ln(2)(\eta(2q)+\zeta(2q)). \tag{4.13}$$

To establish another relation, let $a_n = \frac{\overline{H}_n}{n^{2q}}$ in (1.5):

$$\sum_{n=1}^{\infty}(-1)^n a_n + \sum_{n=1}^{\infty} a_n = 2\sum_{n=1}^{\infty} a_{2n},$$

we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n}{n^{2q}} + \sum_{n=1}^{\infty}\frac{\overline{H}_n}{n^{2q}} &= 2\sum_{n=1}^{\infty}\frac{\overline{H}_{2n}}{(2n)^{2q}}\\ &\{\text{substitute } \overline{H}_{2n} = H_{2n}-H_n \text{ given in (1.137)}\}\\ &= 2\sum_{n=1}^{\infty}\frac{H_{2n}}{(2n)^{2q}} - 2\sum_{n=1}^{\infty}\frac{H_n}{(2n)^{2q}}\\ &\{\text{employ (1.5) for the first sum}\}\\ &= \sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^{2q}} + \sum_{n=1}^{\infty}\frac{H_n}{n^{2q}} - 2\sum_{n=1}^{\infty}\frac{H_n}{(2n)^{2q}}\\ &= \sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^{2q}} + (1-2^{1-2q})\sum_{n=1}^{\infty}\frac{H_n}{n^{2q}}.\end{aligned}$$

Rearrange the terms,

$$\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n}{n^{2q}} - \sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^{2q}} = (1-2^{1-2q})\sum_{n=1}^{\infty}\frac{H_n}{n^{2q}} - \sum_{n=1}^{\infty}\frac{\overline{H}_n}{n^{2q}}. \tag{4.14}$$

Take the difference of (4.13) and (4.14) then divide by 2,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^{2q}} &= -\frac{1}{2}\ln(2)(\zeta(2q)+\eta(2q)) + \frac{1}{2}\sum_{k=1}^{2q-2}(-2)^{-k}\zeta(2q-k)\zeta(k+1)\\ &\quad + \frac{1}{2}\sum_{n=1}^{\infty}\frac{\overline{H}_n}{n^{2q}} - \frac{1-2^{1-2q}}{2}\sum_{n=1}^{\infty}\frac{H_n}{n^{2q}}.\end{aligned}$$

The last two sums are given in (4.9) and (4.5). As for the first, we have

$$\sum_{k=1}^{2q-2}(-2)^{-k}\zeta(2q-k)\zeta(k+1) = \sum_{k=1}^{q-1}[(-2)^{1-2k}+4^{k-q}]\zeta(2q-2k+1)\zeta(2k), \tag{4.15}$$

which follows from using the same technique as in (3.42).

For other proofs, check [4, II.1, pp. 4–7] and [26].

Examples:

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^2}=\frac{5}{8}\zeta(3);$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^4}=\frac{1}{2}\zeta(2)\zeta(3)-\frac{59}{32}\zeta(5);$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^6}=\frac{1}{2}\zeta(2)\zeta(5)+\frac{7}{8}\zeta(3)\zeta(4)-\frac{377}{128}\zeta(7);$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^8}=\frac{1}{2}\zeta(2)\zeta(7)+\frac{31}{32}\zeta(3)\zeta(6)+\frac{7}{8}\zeta(4)\zeta(5)-\frac{2039}{512}\zeta(9).$$

4.1.6 $\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n}{n^{2q}}$

Let $q\in\mathbb{Z}^+$. Then the following identity holds:

$$\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n}{n^{2q}}=\left(q-\frac{2q+1}{2^{2q+1}}\right)\zeta(2q+1)-2\left(1-\frac{1}{2^{2q}}\right)\ln(2)\zeta(2q)$$
$$-\sum_{k=1}^{q-1}\zeta(2k)\eta(2q-2k+1). \tag{4.16}$$

Proof (i).

$$\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n}{n^{q}}=\sum_{n=1}^{\infty}(-1)^n\overline{H}_n\left(\frac{(-1)^{q-1}}{(q-1)!}\int_0^1 x^{n-1}\ln^{q-1}(x)\mathrm{d}x\right)$$
$$=\frac{(-1)^{q-1}}{(q-1)!}\int_0^1\frac{\ln^{q-1}(x)}{x}\left(\sum_{n=1}^{\infty}\overline{H}_n(-x)^n\right)\mathrm{d}x$$

{replace x with $-x$ in (2.29) to get this sum}

$$=\frac{(-1)^{q-1}}{(q-1)!}\int_0^1\frac{\ln^{q-1}(x)}{x}\left(\frac{\ln(1-x)}{1+x}\right)\mathrm{d}x$$
$$=\frac{(-1)^{q}}{(q-1)!}\int_0^1\frac{\ln^{q-1}(x)\ln(1-x)}{x(1+x)}\mathrm{d}x. \tag{4.17}$$

Replace q with $2q$,

$$\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n}{n^{2q}}=\frac{-1}{(2q-1)!}\int_0^1\frac{\ln^{2q-1}(x)\ln(1-x)}{x(1+x)}\mathrm{d}x$$

and this integral is given in (3.54).

Proof (ii). Combine (4.13) and (4.14) then divide by 2,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n}{n^{2q}}&=-\frac{1}{2}\ln(2)(\zeta(2q)+\eta(2q))+\frac{1}{2}\sum_{k=1}^{2q-2}(-2)^{-k}\zeta(2q-k)\zeta(k+1)\\&\quad-\frac{1}{2}\sum_{n=1}^{\infty}\frac{\overline{H}_n}{n^{2q}}+\frac{1-2^{1-2q}}{2}\sum_{n=1}^{\infty}\frac{H_n}{n^{2q}}.\end{aligned}$$

Combine the results (4.15), (4.9), and (4.5) to end the proof.

Examples:

$$\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n}{n^2}=\frac{5}{8}\zeta(3)-\frac{3}{2}\ln(2)\zeta(2);$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n}{n^4}=\frac{59}{32}\zeta(5)-\frac{15}{8}\ln(2)\zeta(4)-\frac{3}{4}\zeta(2)\zeta(3);$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n}{n^6}=\frac{377}{128}\zeta(7)-\frac{63}{32}\ln(2)\zeta(6)-\frac{15}{16}\zeta(2)\zeta(5)-\frac{3}{4}\zeta(3)\zeta(4);$$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n}{n^8}&=\frac{2039}{512}\zeta(9)-\frac{255}{128}\ln(2)\zeta(8)-\frac{63}{64}\zeta(2)\zeta(7)-\frac{3}{4}\zeta(3)\zeta(6)\\&\quad-\frac{15}{16}\zeta(4)\zeta(5).\end{aligned}$$

4.1.7 $\sum_{n=1}^{\infty}\frac{H_{\frac{n}{2}}}{n^{2q}}$

Let $q\in\mathbb{Z}_{>1}$. Then the following identity holds:

$$\sum_{n=1}^{\infty}\frac{H_{\frac{n}{2}}}{n^{2q}}=\left(1+\frac{2q+1}{2^{2q+1}}\right)\zeta(2q+1)-\frac{1}{2^{2q}}\sum_{k=1}^{q-1}2^{2k}\zeta(2q-2k+1)\zeta(2k).\tag{4.18}$$

Proof (i).

$$\sum_{n=1}^{\infty}\frac{H_{\frac{n}{2}}}{n^q}=\sum_{n=1}^{\infty}H_{\frac{n}{2}}\left(\frac{(-1)^{q-1}}{(q-1)!}\int_0^1 x^{n-1}\ln^{q-1}(x)\mathrm{d}x\right)$$
$$=\frac{(-1)^{q-1}}{(q-1)!}\int_0^1\frac{\ln^{q-1}(x)}{x}\left(\sum_{n=1}^{\infty}H_{\frac{n}{2}}x^n\right)\mathrm{d}x$$
$$\{\text{recall the generating function (2.34)}\}$$
$$=\frac{(-1)^{q-1}}{(q-1)!}\int_0^1\frac{\ln^{q-1}(x)}{x}\left(\frac{-2\ln(2)x}{1-x^2}-\frac{2\ln(1-x)}{1-x^2}\right)\mathrm{d}x$$
$$=\frac{2(-1)^q\ln(2)}{(q-1)!}\int_0^1\frac{\ln^{q-1}(x)}{1-x^2}\mathrm{d}x+\frac{(-1)^q}{(q-1)!}\int_0^1\frac{2\ln^{q-1}(x)\ln(1-x)}{x(1-x^2)}\mathrm{d}x$$
$$\left\{\text{write }\frac{2}{x(1-x^2)}=\frac{2}{x}-\frac{1}{1+x}+\frac{1}{1-x}\text{ in the second integral}\right\}$$
$$=\frac{2(-1)^q\ln(2)}{(q-1)!}\int_0^1\frac{\ln^{q-1}(x)}{1-x^2}\mathrm{d}x+\frac{2(-1)^q}{(q-1)!}\int_0^1\frac{\ln^{q-1}(x)\ln(1-x)}{x}\mathrm{d}x$$
$$-\frac{(-1)^q}{(q-1)!}\int_0^1\frac{\ln^{q-1}(x)\ln(1-x)}{1+x}\mathrm{d}x+\frac{(-1)^q}{(q-1)!}\int_0^1\frac{\ln^{q-1}(x)\ln(1-x)}{1-x}\mathrm{d}x.$$

Replace q with $2q$,

$$\sum_{n=1}^{\infty}\frac{H_{\frac{n}{2}}}{n^{2q}}=\frac{2\ln(2)}{(2q-1)!}\int_0^1\frac{\ln^{2q-1}(x)}{1-x^2}\mathrm{d}x+\frac{2}{(2q)!}\int_0^1\frac{\ln^{2q}(x)}{1-x}\mathrm{d}x$$
$$-\frac{1}{(2q-1)!}\int_0^1\frac{\ln^{2q-1}(x)\ln(1-x)}{1+x}\mathrm{d}x+\frac{1}{(2q-1)!}\int_0^1\frac{\ln^{2q-1}(x)\ln(1-x)}{1-x}\mathrm{d}x.$$

These integrals are given in (3.14), (3.8), (3.53), and (3.43) respectively.

Proof (ii). Substitute the results (4.5) and (4.10) in (4.12).

Examples:

$$\sum_{n=1}^{\infty}\frac{H_{\frac{n}{2}}}{n^2}=\frac{11}{8}\zeta(3);$$
$$\sum_{n=1}^{\infty}\frac{H_{\frac{n}{2}}}{n^4}=\frac{37}{32}\zeta(5)-\frac{1}{4}\zeta(2)\zeta(3);$$
$$\sum_{n=1}^{\infty}\frac{H_{\frac{n}{2}}}{n^6}=\frac{135}{128}\zeta(7)-\frac{1}{16}\zeta(2)\zeta(5)-\frac{1}{4}\zeta(3)\zeta(4);$$
$$\sum_{n=1}^{\infty}\frac{H_{\frac{n}{2}}}{n^8}=\frac{521}{512}\zeta(9)-\frac{1}{64}\zeta(2)\zeta(7)-\frac{1}{4}\zeta(3)\zeta(6)-\frac{1}{16}\zeta(4)\zeta(5).$$

4.1.8 $\sum_{n=1}^{\infty}\frac{(-1)^n H_{\frac{n}{2}}}{n^{2q}}$

Let $q \in \mathbb{Z}^+$. Then the following identity holds:

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_{\frac{n}{2}}}{n^{2q}} = \left(\frac{2q+3}{2^{2q+1}}-1\right)\zeta(2q+1)$$
$$-\sum_{k=1}^{q-1}(2^{1-2q}-2^{2k-2q})\zeta(2q-2k+1)\zeta(2k). \tag{4.19}$$

Proof (i).

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_{\frac{n}{2}}}{n^{q}} = \sum_{n=1}^{\infty}(-1)^n H_{\frac{n}{2}}\left(\frac{(-1)^{q-1}}{(q-1)!}\int_0^1 x^{n-1}\ln^{q-1}(x)\mathrm{d}x\right)$$
$$= \frac{(-1)^{q-1}}{(q-1)!}\int_0^1 \frac{\ln^{q-1}(x)}{x}\left(\sum_{n=1}^{\infty}H_{\frac{n}{2}}(-x)^n\right)\mathrm{d}x$$

{replace x with $-x$ in the generating function (2.34)}

$$= \frac{(-1)^{q-1}}{(q-1)!}\int_0^1 \frac{\ln^{q-1}(x)}{x}\left(\frac{2\ln(2)x}{1-x^2}-\frac{2\ln(1+x)}{1-x^2}\right)\mathrm{d}x$$
$$= -\frac{2(-1)^q\ln(2)}{(q-1)!}\int_0^1\frac{\ln^{q-1}(x)}{1-x^2}\mathrm{d}x+\frac{(-1)^q}{(q-1)!}\int_0^1\frac{2\ln^{q-1}(x)\ln(1+x)}{x(1-x^2)}\mathrm{d}x$$

$$\left\{\text{write } \frac{2x}{1-x^2}=\frac{1}{x}+\frac{1}{x(1+x)}+\frac{1}{1-x} \text{ in the second integral}\right\}$$

$$= -\frac{2(-1)^q\ln(2)}{(q-1)!}\int_0^1\frac{\ln^{q-1}(x)}{1-x^2}\mathrm{d}x+\frac{(-1)^q}{(q-1)!}\int_0^1\frac{\ln^{q-1}(x)\ln(1+x)}{x}\mathrm{d}x$$
$$+\frac{(-1)^q}{(q-1)!}\int_0^1\frac{\ln^{q-1}(x)\ln(1+x)}{x(1+x)}\mathrm{d}x+\frac{(-1)^q}{(q-1)!}\int_0^1\frac{\ln^{q-1}(x)\ln(1+x)}{1-x}\mathrm{d}x. \tag{4.20}$$

Replace q with $2q$,

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_{\frac{n}{2}}}{n^{2q}}$$
$$= -\frac{2\ln(2)}{(2q-1)!}\int_0^1\frac{\ln^{2q-1}(x)}{1-x^2}\mathrm{d}x+\frac{1}{(2q-1)!}\int_0^1\frac{\ln^{2q-1}(x)\ln(1+x)}{x}\mathrm{d}x$$
$$+\frac{1}{(2q-1)!}\int_0^1\frac{\ln^{2q-1}(x)\ln(1+x)}{x(1+x)}\mathrm{d}x+\frac{1}{(2q-1)!}\int_0^1\frac{\ln^{2q-1}(x)\ln(1+x)}{1-x}\mathrm{d}x.$$

These integrals are given in (3.14), (3.12), (3.36), and (3.43) respectively.

Proof (ii). Set $a_n = H_{\frac{n}{2}}/n^{2q}$ in (1.5):

$$\sum_{n=1}^{\infty}(-1)^n a_n = 2\sum_{n=1}^{\infty}a_{2n} - \sum_{n=1}^{\infty}a_n,$$

we get

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_{\frac{n}{2}}}{n^{2q}} = 2^{1-2q}\sum_{n=1}^{\infty}\frac{H_n}{n^{2q}} - \sum_{n=1}^{\infty}\frac{H_{\frac{n}{2}}}{n^{2q}}.$$

These two sums are found in (4.5) and (4.18).

Examples:

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_{\frac{n}{2}}}{n^2} = -\frac{3}{8}\zeta(3);$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_{\frac{n}{2}}}{n^4} = \frac{1}{8}\zeta(2)\zeta(3) - \frac{25}{32}\zeta(5);$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_{\frac{n}{2}}}{n^6} = \frac{1}{32}\zeta(2)\zeta(5) + \frac{7}{32}\zeta(3)\zeta(4) - \frac{119}{128}\zeta(7);$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_{\frac{n}{2}}}{n^8} = \frac{1}{128}\zeta(2)\zeta(7) + \frac{31}{128}\zeta(3)\zeta(6) + \frac{7}{128}\zeta(4)\zeta(5) - \frac{501}{512}\zeta(9).$$

4.1.9 $\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^{2q-1}}$

Let $q \in \mathbb{Z}_{>1}$. Then the following identity holds:

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^{2q-1}} = \zeta(2)\zeta(2q-1) - \frac{1}{2}(q+1)(2q-1)\zeta(2q+1) + 2\sum_{k=1}^{q-1}k\,\zeta(2q-2k)\zeta(2k+1). \tag{4.21}$$

Proof. First, let's find the following integral:

$$\int_0^1 x^{n-1}\ln(x)\ln(1-x)\mathrm{d}x = \frac{d}{dn}\int_0^1 x^{n-1}\ln(1-x)\mathrm{d}x$$

{recall the integral (1.124)}

$$= \frac{d}{dn}\left(-\frac{H_n}{n}\right)$$

{use the derivative of the harmonic number (1.132)}

$$= \frac{H_n}{n^2} + \frac{H_n^{(2)}}{n} - \frac{\zeta(2)}{n}. \tag{4.22}$$

Next, divide both sides by n^a then take the summation over $n \geq 1$,

$$\sum_{n=1}^{\infty} \frac{H_n}{n^{a+2}} + \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^{a+1}} - \zeta(2)\zeta(a+1)$$

$$= \int_0^1 \frac{\ln(x)\ln(1-x)}{x} \left(\sum_{n=1}^{\infty} \frac{x^n}{n^a}\right) \mathrm{d}x$$

$$= \int_0^1 \frac{\ln(x)\ln(1-x)\operatorname{Li}_a(x)}{x} \mathrm{d}x$$

{expand $\ln(1-x)$ in series}

$$= -\sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 x^{n-1} \ln(x) \operatorname{Li}_a(x) \mathrm{d}x$$

$$= -\sum_{n=1}^{\infty} \frac{1}{n} \frac{d}{dn} \int_0^1 x^{n-1} \operatorname{Li}_a(x) \mathrm{d}x$$

{recall the result (3.65)}

$$= -\sum_{n=1}^{\infty} \frac{1}{n} \frac{d}{dn} \left[(-1)^{a-1} \frac{H_n}{n^a} - \sum_{k=1}^{a-1} (-1)^k \frac{\zeta(a-k+1)}{n^k}\right]$$

$$= -\sum_{n=1}^{\infty} \frac{1}{n} \left[(-1)^a \left(\frac{H_n^{(2)}}{n^a} + \frac{aH_n}{n^{a+1}} - \frac{\zeta(2)}{n^a}\right) + \sum_{k=1}^{a-1} k(-1)^k \frac{\zeta(a-k+1)}{n^{k+1}}\right]$$

$$= -(-1)^a \left(\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^{a+1}} + a\sum_{n=1}^{\infty} \frac{H_n}{n^{a+2}} - \zeta(2)\zeta(a+1)\right)$$

$$- \sum_{k=1}^{a-1} k(-1)^k \zeta(a-k+1)\zeta(k+2).$$

Replace a with $2q-2$ then reorder the terms,

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^{2q-1}} = \zeta(2)\zeta(2q-1) - \frac{1}{2}(2q-1) \sum_{n=1}^{\infty} \frac{H_n}{n^{2q}}$$

$$- \frac{1}{2} \sum_{k=1}^{2q-3} k(-1)^k \zeta(2q-k-1)\zeta(k+2). \tag{4.23}$$

The first sum is given in (4.5). For the second sum, note that the index k can start from zero, since the term $(k=0)$ is zero:

$$\sum_{k=1}^{2q-3} k(-1)^k\zeta(2q-k-1)\zeta(k+2) = \sum_{k=0}^{2q-3} k(-1)^k\zeta(2q-k-1)\zeta(k+2)$$

$$\left\{\text{use} \sum_{k=0}^{2q-3} f(k) = \sum_{k=1}^{q-1} f(2k-2) + \sum_{k=1}^{q-1} f(2k-1)\right\}$$

$$= \sum_{k=1}^{q-1}(2k-2)\zeta(2q-2k+1)\zeta(2k) - \sum_{k=1}^{q-1}(2k-1)\zeta(2q-2k)\zeta(2k+1)$$

$$\{\text{reverse the terms order of the second sum}\}$$

$$= \sum_{k=1}^{q-1}(2k-2)\zeta(2q-2k+1)\zeta(2k) - \sum_{k=1}^{q-1}(2q-2k-1)\zeta(2k)\zeta(2q-2k+1)$$

$$= \sum_{k=1}^{q-1}(4k-2q-1)\zeta(2q-2k+1)\zeta(2k). \tag{4.24}$$

Therefore, our sum simplifies to

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^{2q-1}} = \zeta(2)\zeta(2q-1) - \frac{1}{2}(q+1)(2q-1)\zeta(2q+1)$$
$$+2\sum_{k=1}^{q-1}(q-k)\zeta(2q-2k+1)\zeta(2k).$$

The proof completes on writing

$$\sum_{k=1}^{q-1}(q-k)\zeta(2q-2k+1)\zeta(2k) = \sum_{k=1}^{q-1} k\,\zeta(2q-2k)\zeta(2k+1),$$

which follows from reversing the terms order.

Examples:

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3} = 3\zeta(2)\zeta(3) - \frac{9}{2}\zeta(5);$$

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^5} = 5\zeta(2)\zeta(5) + 2\zeta(3)\zeta(4) - 10\zeta(7).;$$

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^7} = 7\zeta(2)\zeta(7) + 2\zeta(3)\zeta(6) + 4\zeta(4)\zeta(5) - \frac{35}{2}\zeta(9);$$

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^9} = 9\zeta(2)\zeta(9) + 2\zeta(3)\zeta(8) + 6\zeta(4)\zeta(7) + 4\zeta(5)\zeta(6) - 27\zeta(11).$$

4.1.10 $\sum_{n=1}^{\infty} \frac{H_n^{(2q-1)}}{n^2}$

Let $q \in \mathbb{Z}_{>1}$. Then the following identity holds:

$$\sum_{n=1}^{\infty} \frac{H_n^{(2q-1)}}{n^2} = \left(1 + \frac{1}{2}(q+1)(2q-1)\right)\zeta(2q+1) \\ -2\sum_{k=1}^{q-1} k\,\zeta(2q-2k)\zeta(2k+1). \tag{4.25}$$

Proof. Set $p = 2q - 1$ and $q = 2$ in (2.76):

$$\sum_{k=1}^{\infty} \frac{H_k^{(p)}}{k^q} + \sum_{k=1}^{\infty} \frac{H_k^{(q)}}{k^p} = \zeta(p)\zeta(q) + \zeta(p+q),$$

we have

$$\sum_{n=1}^{\infty} \frac{H_n^{(2q-1)}}{n^2} + \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^{2q-1}} = \zeta(2q-1)\zeta(2) + \zeta(2q+1).$$

Substituting the result (4.21) finishes the proof.

Examples:

$$\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^2} = \frac{11}{2}\zeta(5) - 2\zeta(2)\zeta(3);$$

$$\sum_{n=1}^{\infty} \frac{H_n^{(5)}}{n^2} = 11\zeta(7) - 4\zeta(2)\zeta(5) - 2\zeta(3)\zeta(4);$$

$$\sum_{n=1}^{\infty} \frac{H_n^{(7)}}{n^2} = \frac{37}{2}\zeta(9) - 6\zeta(2)\zeta(7) - 2\zeta(3)\zeta(6) - 4\zeta(4)\zeta(5);$$

$$\sum_{n=1}^{\infty} \frac{H_n^{(9)}}{n^2} = 28\zeta(11) - 8\zeta(2)\zeta(9) - 2\zeta(3)\zeta(8) - 6\zeta(4)\zeta(7) - 4\zeta(5)\zeta(6).$$

4.1.11 $\sum_{n=1}^{\infty} \frac{H_n^2}{n^{2q-1}}$

Let $q \in \mathbb{Z}_{>1}$. Then the following identity holds:

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^{2q-1}} = \frac{1}{3} \sum_{k=1}^{2q-3} \sum_{j=1}^{k-1} (-1)^k \zeta(2q-k-1)\zeta(k-j+1)\zeta(j+1)$$
$$-\frac{1}{3} \sum_{k=1}^{q-1} (2k+1)\zeta(2k)\zeta(2q-2k+1) + \frac{(q+1)(2q+3)}{6}\zeta(2q+1). \quad (4.26)$$

Proof.

$$\sum_{n=1}^{\infty} \frac{H_n^2 + H_n^{(2)}}{n^{2q-1}} = \sum_{n=1}^{\infty} \frac{1}{n^{2n-2}} \left(\frac{H_n^2 + H_n^{(2)}}{n} \right)$$

{use the integral (2.62)}

$$= \sum_{n=1}^{\infty} \frac{1}{n^{2n-2}} \left(\int_0^1 x^{n-1} \ln^2(1-x) \mathrm{d}x \right)$$
$$= \int_0^1 \frac{\ln^2(1-x)}{x} \left(\sum_{n=1}^{\infty} \frac{x^n}{n^{2q-2}} \right) \mathrm{d}x$$
$$= \int_0^1 \frac{\ln^2(1-x) \operatorname{Li}_{2q-2}(x)}{x} \mathrm{d}x$$

{expand $\ln^2(1-x)$ in series given in (2.7)}

$$= 2 \sum_{n=1}^{\infty} \frac{H_{n-1}}{n} \int_0^1 x^{n-1} \operatorname{Li}_{2q-2}(x) \mathrm{d}x$$

{recall the result (3.65)}

$$= 2 \sum_{n=1}^{\infty} \frac{H_n - \frac{1}{n}}{n} \left(-\frac{H_n}{n^{2q-2}} - \sum_{k=1}^{2q-3} (-1)^k \frac{\zeta(2q-k-1)}{n^k} \right).$$

Distributing and rearranging the terms,

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^{2q-1}} = \frac{2}{3} \sum_{n=1}^{\infty} \frac{H_n}{n^{2q}} - \frac{1}{3} \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^{2q-1}}$$
$$+ \frac{2}{3} \sum_{k=1}^{2q-3} (-1)^k \zeta(2q-k-1) \left(\zeta(k+2) - \sum_{n=1}^{\infty} \frac{H_n}{n^{k+1}} \right).$$

Recall the relation involving $\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^{2q-1}}$ from (4.23),

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{H_n^2}{n^{2q-1}} &= \frac{2q+3}{6} \sum_{n=1}^{\infty} \frac{H_n}{n^{2q}} - \frac{1}{3}\zeta(2)\zeta(2q-1) \\
&+ \frac{1}{3} \sum_{k=1}^{2q-3} (-1)^k \zeta(2q-k-1) \left(\frac{k+4}{2}\zeta(k+2) - 2\sum_{n=1}^{\infty} \frac{H_n}{n^{k+1}} \right) \\
&\quad \{\text{substitute the Euler sums (4.5) and (4.4)}\} \\
&= \frac{1}{6}(q+1)(2q+3)\zeta(2q+1) - \frac{1}{3}\zeta(2)\zeta(2q-1) \\
&+ \frac{1}{3} \sum_{k=1}^{2q-3} \sum_{j=1}^{k-1} (-1)^k \zeta(2q-k-1)\zeta(k-j+1)\zeta(j+1) \\
&- \frac{1}{6} \sum_{k=1}^{2q-3} (-1)^k (k+2)\zeta(k+2)\zeta(2q-k-1) \\
&- \frac{2q+3}{6} \sum_{k=1}^{q-1} \zeta(2q-2k+1)\zeta(2k).
\end{aligned}$$

The proof completes on writing

$$\begin{aligned}
&\sum_{k=1}^{2q-3} (-1)^k (k+2)\zeta(k+2)\zeta(2q-k-1) \\
&= -2\zeta(2)\zeta(2q-1) + \sum_{k=1}^{q-1} (4k-2q-1)\zeta(2k)\zeta(2q-2k+1),
\end{aligned}$$

which follows from adding and subtracting the term $(k=0)$ then using the same idea in (4.24).

Examples:

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^3} = \frac{7}{2}\zeta(5) - \zeta(2)\zeta(3);$$

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^5} = 6\zeta(7) - \zeta(2)\zeta(5) - \frac{5}{2}\zeta(3)\zeta(4);$$

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^7} = \frac{55}{6}\zeta(9) - \zeta(2)\zeta(7) - \frac{7}{2}\zeta(3)\zeta(6) + \frac{1}{3}\zeta^3(3) - \frac{5}{2}\zeta(4)\zeta(5);$$

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^9} = 11\zeta(11) - \zeta(2)\zeta(9) - \frac{9}{2}\zeta(3)\zeta(8) - \frac{5}{2}\zeta(4)\zeta(7) - \frac{7}{2}\zeta(5)\zeta(6)$$

$$+\zeta^2(3)\zeta(5).$$

4.1.12 $\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(q)}}{n}$

Let $q \in \mathbb{Z}_{>1}$. Then the following identity holds:

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(q)}}{n} = \ln(2)\eta(q) - \frac{q}{2}\zeta(q+1) + \frac{1}{2}\sum_{j=1}^{q-2}\eta(q-j)\eta(j+1). \tag{4.27}$$

The following proof is by Cornel Vălean and may be found in [29]:

Proof (i). Using the integral form of $H_n^{(q)}$ given in (1.122), we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(q)}}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left(\frac{(-1)^{q-1}}{(q-1)!}\int_0^1 \frac{\ln^{q-1}(x)(1-x^n)}{1-x}\mathrm{d}x\right)$$

$$= \frac{(-1)^{q-1}}{(q-1)!}\int_0^1 \frac{\ln^{q-1}(x)}{1-x}\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n} - \sum_{n=1}^{\infty}\frac{(-x)^n}{n}\right)\mathrm{d}x$$

$$= \frac{(-1)^{q-1}}{(q-1)!}\int_0^1 \frac{\ln^{q-1}(x)}{1-x}(-\ln(2)+\ln(1+x))\mathrm{d}x$$

$$= \frac{(-1)^{q}}{(q-1)!}\int_0^1 \frac{\ln^{q-1}(x)\ln\left(\frac{2}{1+x}\right)}{1-x}\mathrm{d}x. \tag{4.28}$$

This integral is given in (3.47).

Proof (ii). We continue what we reached in (4.28),

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(q)}}{n} = \frac{(-1)^{q}}{(q-1)!}\int_0^1 \frac{\ln^{q-1}(x)}{1-x}(\ln(2)-\ln(1+x))\mathrm{d}x$$

$$= \frac{(-1)^{q}}{(q-1)!}\ln(2)\int_0^1 \frac{\ln^{q-1}(x)}{1-x}\mathrm{d}x - \frac{(-1)^{q}}{(q-1)!}\int_0^1 \frac{\ln^{q-1}(x)\ln(1+x)}{1-x}\mathrm{d}x$$

$$\left\{\text{write } \frac{\ln(1+x)}{1-x} = \sum_{n=1}^{\infty}\overline{H}_n x^n \text{ given in (2.29)}\right\}$$

$$= -\ln(2)\zeta(q) - \frac{(-1)^{q}}{(q-1)!}\sum_{n=1}^{\infty}\overline{H}_n\int_0^1 x^n\ln^{q-1}(x)\mathrm{d}x$$

$$= -\ln(2)\zeta(q) + \sum_{n=1}^{\infty}\frac{\overline{H}_n}{(n+1)^q}$$

$$\{\text{let the index start from } 0 \text{ then shift the index by } -1\}$$

$$= -\ln(2)\zeta(q) + \sum_{n=1}^{\infty} \frac{\overline{H}_{n-1}}{n^q}$$

$$\left\{\text{use } \overline{H}_{n-1} = \overline{H}_n + \frac{(-1)^n}{n} \text{ given in (1.136)}\right\}$$

$$= -\ln(2)\zeta(q) + \sum_{n=1}^{\infty} \frac{\overline{H}_n}{n^q} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{q+1}}$$

$$= -\ln(2)\zeta(q) + \sum_{n=1}^{\infty} \frac{\overline{H}_n}{n^q} - \eta(q+1).$$

Collect the result (4.7) and use $\eta(a) = (1 - 2^{1-a})\zeta(a)$ to finish the proof.

Examples:

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(2)}}{n} = \frac{1}{2}\ln(2)\zeta(2) - \zeta(3);$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(3)}}{n} = \frac{3}{4}\ln(2)\zeta(3) - \frac{19}{16}\zeta(4);$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(4)}}{n} = \frac{7}{8}\ln(2)\zeta(4) + \frac{3}{8}\zeta(2)\zeta(3) - 2\zeta(5);$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(5)}}{n} = \frac{15}{16}\ln(2)\zeta(5) + \frac{9}{32}\zeta^2(3) - \frac{111}{64}\zeta(6).$$

4.1.13 $\sum_{n=1}^{\infty} \frac{\zeta(q)-H_n^{(q)}}{n}$

Let $q \in \mathbb{Z}_{>1}$. Then the following identity holds:

$$\sum_{n=1}^{\infty} \frac{\zeta(q) - H_n^{(q)}}{n} = \frac{q}{2}\zeta(q+1) - \frac{1}{2}\sum_{j=1}^{q-2} \zeta(q-j)\zeta(j+1). \tag{4.29}$$

Proof (i). Employ the integral form of $\zeta(q) - H_n^{(q)}$ form given in (1.126),

$$\sum_{n=1}^{\infty} \frac{\zeta(q) - H_n^{(q)}}{n} = \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{(-1)^{q-1}}{(q-1)!}\int_0^1 \frac{x^n \ln^{q-1}(x)}{1-x}\mathrm{d}x\right)$$

$$= \frac{(-1)^{q-1}}{(q-1)!} \int_0^1 \frac{\ln^{q-1}(x)}{1-x} \left(\sum_{n=1}^{\infty} \frac{x^n}{n} \right) \mathrm{d}x$$
$$= \frac{(-1)^{q}}{(q-1)!} \int_0^1 \frac{\ln^{q-1}(x)\ln(1-x)}{1-x} \mathrm{d}x.$$

This integral is given in (3.41).

Proof (ii). Let $b_k = \zeta(q) - H_k^{(q)}$ and $a_k = \frac{1}{k}$ in (2.75):

$$\sum_{k=1}^{n} a_k b_k = A_n b_n - \sum_{k=0}^{n-1} A_k \left(b_{k+1} - b_k\right), \quad A_n = \sum_{i=1}^{n} a_i,$$

we have

$$\sum_{k=1}^{n} \frac{\zeta(q) - H_k^{(q)}}{k}$$
$$= (\zeta(q) - H_n^{(q)}) \sum_{i=1}^{n} \frac{1}{i} - \sum_{k=0}^{n-1} \left(\sum_{i=1}^{k} \frac{1}{i} \right) \left(-H_{k+1}^{(q)} + H_k^{(q)} \right)$$
$$= (\zeta(q) - H_n^{(q)}) H_n - \sum_{k=0}^{n-1} (H_k) \left(\frac{-1}{(k+1)^q} \right).$$

Let $n \mapsto \infty$ and use $\lim_{n\to\infty} H_n^{(q)} = \zeta(q)$,

$$\sum_{k=1}^{\infty} \frac{\zeta(q) - H_k^{(q)}}{k} = 0 + \sum_{k=0}^{\infty} \frac{H_k}{(k+1)^q}$$
$$\{\text{shift the index } k \text{ by } -1\}$$
$$= \sum_{k=1}^{\infty} \frac{H_{k-1}}{k^q}$$
$$= \sum_{k=1}^{\infty} \frac{H_k - \frac{1}{n}}{k^q}$$
$$= \sum_{k=1}^{\infty} \frac{H_k}{k^q} - \zeta(q+1).$$

Collect the result (4.4) to finish the proof.

Examples:

$$\sum_{n=1}^{\infty} \frac{\zeta(2) - H_n^{(2)}}{n} = \zeta(3);$$

$$\sum_{n=1}^{\infty} \frac{\zeta(3) - H_n^{(3)}}{n} = \frac{1}{4}\zeta(4);$$

$$\sum_{n=1}^{\infty} \frac{\zeta(4) - H_n^{(4)}}{n} = 2\zeta(5) - \zeta(2)\zeta(3);$$

$$\sum_{n=1}^{\infty} \frac{\zeta(5) - H_n^{(5)}}{n} = \frac{3}{4}\zeta(6) - \frac{1}{2}\zeta^2(3).$$

4.1.14 $\sum_{n=1}^{\infty} \frac{H_n}{(2n+1)^q}$

Let $q \in \mathbb{Z}_{>1}$. Then the following identity holds:

$$\sum_{n=1}^{\infty} \frac{H_n}{(2n+1)^q} = 2\left(\frac{1}{2^q} - 1\right)\ln(2)\zeta(q) + q\left(1 - \frac{1}{2^{q+1}}\right)\zeta(q+1)$$
$$-\frac{1}{2}\sum_{j=1}^{q-2}(2^{j+1} - 1)(2^{-j} - 2^{-q})\zeta(q-j)\zeta(j+1). \tag{4.30}$$

Proof (i).

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{H_n}{(2n+1)^q} &= \sum_{n=1}^{\infty} H_n \left(\frac{(-1)^{q-1}}{(q-1)!}\int_0^1 x^{2n}\ln^{q-1}(x)\mathrm{d}x\right) \\
&= \frac{(-1)^{q-1}}{(a-1)!}\int_0^1 \ln^{q-1}(x)\left(\sum_{n=1}^{\infty} H_n x^{2n}\right)\mathrm{d}x \\
&= \frac{(-1)^{q-1}}{(a-1)!}\int_0^1 \ln^{q-1}(x)\left(\frac{-\ln(1-x^2)}{1-x^2}\right)\mathrm{d}x \\
&= \frac{(-1)^{q}}{(q-1)!}\int_0^1 \frac{\ln^{q-1}(x)\ln(1-x^2)}{1-x^2}\mathrm{d}x.
\end{aligned} \tag{4.31}$$

This integral is given in (3.45).

Proof (ii). Set $a_n = \frac{\overline{H}_n}{n^q}$ in (1.4):

$$\sum_{n=1}^{\infty} a_n = \sum_{n=0}^{\infty} a_{2n+1} + \sum_{n=1}^{\infty} a_{2n},$$

we obtain

$$\sum_{n=1}^{\infty}\frac{\overline{H}_n}{n^q}=\sum_{n=0}^{\infty}\frac{\overline{H}_{2n+1}}{(2n+1)^q}+\sum_{n=1}^{\infty}\frac{\overline{H}_{2n}}{(2n)^q}.$$

For the first sum, write $\overline{H}_{2n+1}=H_{2n+1}-H_n$ given in (1.138),

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\overline{H}_{2n+1}}{(2n+1)^q}&=\sum_{n=0}^{\infty}\frac{H_{2n+1}}{(2n+1)^q}-\sum_{n=0}^{\infty}\frac{H_n}{(2n+1)^q}\\
&\{\text{make use of (1.5) for the first sum}\}\\
&\{\text{and let the index start from 1 in the second sum, since } H_0=0\}\\
&=\frac{1}{2}\sum_{n=0}^{\infty}\frac{H_{n+1}}{(n+1)^q}+\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^nH_{n+1}}{(n+1)^q}-\sum_{n=1}^{\infty}\frac{H_n}{(2n+1)^q}\\
&\{\text{shift the index } n \text{ of the first and second sums by } -1\}\\
&=\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n}{n^q}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^q}-\sum_{n=1}^{\infty}\frac{H_n}{(2n+1)^q}.\end{aligned}$$

For the second sum, write $\overline{H}_{2n}=H_{2n}-H_n$ given in (1.137),

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\overline{H}_{2n}}{(2n)^q}&=\sum_{n=1}^{\infty}\frac{H_{2n}}{(2n)^q}-\sum_{n=1}^{\infty}\frac{H_n}{(2n)^q}\\
&\{\text{make use of (1.5) for the first sum}\}\\
&=\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n}{n^q}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^q}-2^{-q}\sum_{n=1}^{\infty}\frac{H_n}{n^q}\\
&=\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^q}+\left(\frac{1}{2}-\frac{1}{2^q}\right)\sum_{n=1}^{\infty}\frac{H_n}{n^q}.\end{aligned}$$

Combining the two sums, we arrive at

$$\sum_{n=1}^{\infty}\frac{H_n}{(2n+1)^q}=\left(1-\frac{1}{2^q}\right)\sum_{n=1}^{\infty}\frac{H_n}{n^q}-\sum_{n=1}^{\infty}\frac{\overline{H}_n}{n^q}$$

and the proof follows on collecting the results (4.4) and (4.7).

Examples:

$$\sum_{n=1}^{\infty}\frac{H_n}{(2n+1)^2}=\frac{7}{4}\zeta(3)-\frac{3}{2}\ln(2)\zeta(2);$$

$$\sum_{n=1}^{\infty}\frac{H_n}{(2n+1)^3}=\frac{45}{32}\zeta(4)-\frac{7}{4}\ln(2)\zeta(3);$$

$$\sum_{n=1}^{\infty}\frac{H_n}{(2n+1)^4}=\frac{31}{8}\zeta(5)-\frac{15}{8}\ln(2)\zeta(4)-\frac{21}{16}\zeta(2)\zeta(3);$$

$$\sum_{n=1}^{\infty}\frac{H_n}{(2n+1)^5}=\frac{315}{128}\zeta(6)-\frac{31}{16}\ln(2)\zeta(5)-\frac{49}{64}\zeta^2(3).$$

4.1.15 $\sum_{n=1}^{\infty}\frac{\overline{H}_n}{(2n+1)^q}$

Let $q\in\mathbb{Z}_{>1}$. Then the following identity holds:

$$\sum_{n=1}^{\infty}\frac{\overline{H}_n}{(2n+1)^q}=\left(1-\frac{1}{2^q}\right)\ln(2)\zeta(q)-q\left(1-\frac{1}{2^{q+1}}\right)\zeta(q+1)+\sum_{k=0}^{q-1}\beta(q-k)\beta(k+1). \tag{4.32}$$

The following proof is due to Sean Stewart and it may be found in [23, p. 92 – 95]:
Proof.

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\overline{H}_n}{(2n+1)^q}&=\sum_{n=1}^{\infty}\overline{H}_n\left(\frac{(-1)^{q-1}}{(q-1)!}\int_0^1 x^{2n}\ln^{q-1}(x)\mathrm{d}x\right)\\&=\frac{(-1)^{q-1}}{(q-1)!}\int_0^1\ln^{q-1}(x)\left(\sum_{n=1}^{\infty}\overline{H}_n x^{2n}\right)\mathrm{d}x\\&=\frac{(-1)^{q-1}}{(q-1)!}\int_0^1\ln^{q-1}(x)\left(\frac{\ln(1+x^2)}{1-x^2}\right)\mathrm{d}x.\end{aligned}$$

This integral is given in (3.63).
Examples:

$$\sum_{n=1}^{\infty}\frac{\overline{H}_n}{(2n+1)^2}=\frac{3}{4}\ln(2)\zeta(2)-\frac{7}{4}\zeta(3)+\frac{\pi}{2}G;$$

$$\sum_{n=1}^{\infty}\frac{\overline{H}_n}{(2n+1)^3}=\frac{7}{8}\ln(2)\zeta(3)-\frac{45}{32}\zeta(4)+G^2;$$

$$\sum_{n=1}^{\infty}\frac{\overline{H}_n}{(2n+1)^4}=\frac{15}{16}\ln(2)\zeta(4)-\frac{31}{8}\zeta(5)+2G\beta(3)+\frac{\pi}{2}\beta(4).$$

$$\sum_{n=1}^{\infty}\frac{\overline{H}_n}{(2n+1)^5}=\frac{31}{32}\ln(2)\zeta(5)-\frac{4095}{1024}\zeta(6)+2G\beta(4)+\frac{\pi}{2}\beta(5).$$

4.1.16 $\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n}{(2n+1)^{2q+1}}$

Let $q\in\mathbb{Z}_{\geq 0}$. Then the following identity holds:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n}{(2n+1)^{2q+1}}&=\ln(2)\beta(2q+1)-(2q+1)\beta(2q+2)\\&+2\sum_{k=0}^{q}(1-2^{2k-2q})\zeta(2q-2k)\beta(2k+2).\end{aligned}\tag{4.33}$$

Proof.

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n}{(2n+1)^{2q+1}}&=\sum_{n=1}^{\infty}(-1)^n\overline{H}_n\left(\frac{1}{(2q)!}\int_0^1 x^{2n}\ln^{2q}(x)\mathrm{d}x\right)\\&=\frac{1}{(2q)!}\int_0^1\ln^{2q}(x)\left(\sum_{n=1}^{\infty}\overline{H}_n(-x^2)^n\right)\mathrm{d}x\\&=\frac{1}{(2q)!}\int_0^1\ln^{2q}(x)\left(\frac{\ln(1-x^2)}{1+x^2}\right)\mathrm{d}x.\end{aligned}$$

This integral is given in (3.58).

For another proof with a different closed form, see [23, p. 92–95].

Examples:

$$\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n}{2n+1}=\frac{\pi}{4}\ln(2)-G;$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n}{(2n+1)^3}=\frac{3}{2}G\zeta(2)+\ln(2)\beta(3)-3\beta(4);$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n}{(2n+1)^5}=\frac{15}{8}G\zeta(4)+\frac{3}{2}\zeta(2)\beta(4)+\ln(2)\beta(5)-5\beta(6);$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n}{(2n+1)^7}=\frac{63}{32}G\zeta(6)+\frac{45}{24}\zeta(4)\beta(4)+\frac{3}{2}\zeta(2)\beta(6)+\ln(2)\beta(7)-7\beta(8).$$

4.1.17 $\sum_{n=1}^{\infty} \frac{\overline{H}_{2n}}{(2n+1)^{2q}}$

Let $q \in \mathbb{Z}^+$. Then the following identity holds:

$$\sum_{n=1}^{\infty} \frac{\overline{H}_{2n}}{(2n+1)^{2q}} = \frac{1}{2}(2q+1)\left(\frac{1}{2^{2q+1}}-1\right)\zeta(2q+1)$$

$$+2\left(1-\frac{1}{2^{2q}}\right)\ln(2)\zeta(2q) + \sum_{k=1}^{q-1}\left(1-\frac{1}{2^{2k}}\right)\zeta(2k)\eta(2q-2k+1). \quad (4.34)$$

Proof. Using

$$\sum_{n=1}^{\infty} a_{2n} = \frac{1}{2}\sum_{n=1}^{\infty}(1+(-1)^n)a_n$$

we have

$$\sum_{n=1}^{\infty} \frac{\overline{H}_{2n}}{(2n+1)^{2q}} = \frac{1}{2}\sum_{n=1}^{\infty}(1+(-1)^n)\frac{\overline{H}_n}{(n+1)^{2q}}$$

$$\{\text{shift the index by } -1\}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}(1-(-1)^n)\frac{\overline{H}_{n-1}}{n^{2q}}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}(1-(-1)^n)\frac{\overline{H}_n + \frac{(-1)^n}{n}}{n^{2q}}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\frac{\overline{H}_n}{n^{2q}} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n}{n^{2q}} - \frac{1}{2}\eta(2q+1) - \frac{1}{2}\zeta(2q+1).$$

These two sums are given in (4.9) and (4.16) respectively.

Examples:

$$\sum_{n=1}^{\infty} \frac{\overline{H}_{2n}}{(2n+1)^{2}} = \frac{3}{2}\ln(2)\zeta(2) - \frac{21}{16}\zeta(3);$$

$$\sum_{n=1}^{\infty} \frac{\overline{H}_{2n}}{(2n+1)^{4}} = \frac{45}{24}\ln(2)\zeta(4) + \frac{9}{16}\zeta(2)\zeta(3) - \frac{155}{64}\zeta(5);$$

$$\sum_{n=1}^{\infty} \frac{\overline{H}_{2n}}{(2n+1)^{6}} = \frac{63}{32}\ln(2)\zeta(6) + \frac{45}{64}\zeta(3)\zeta(4) + \frac{45}{64}\zeta(2)\zeta(5) - \frac{889}{256}\zeta(7);$$

$$\sum_{n=1}^{\infty} \frac{\overline{H}_{2n}}{(2n+1)^{8}} = \frac{255}{128}\ln(2)\zeta(8) + \frac{189}{256}\zeta(3)\zeta(6) + \frac{225}{256}\zeta(4)\zeta(5) + \frac{189}{256}\zeta(2)\zeta(7)$$

$$-\frac{4599}{1024}\zeta(9).$$

4.1.18 $\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(2q+1)}}{2n+1}$

Let $q \in \mathbb{Z}_{\geq 0}$. Then the following identity holds:

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(2q+1)}}{2n+1} = -2^{2p-1}\pi\,\eta(2q+1) + 2\sum_{k=0}^{q} 2^{2k}\eta(2q-2k)\beta(2k+2). \tag{4.35}$$

Proof.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(2q+1)}}{2n+1} &= \sum_{n=1}^{\infty} (-1)^n H_n^{(2q+1)} \int_0^1 x^{2n}\mathrm{d}x \\
&= \int_0^1 \left(\sum_{n=1}^{\infty} H_n^{(2q+1)} \left(-x^2\right)^n \right) \mathrm{d}x \\
&\quad \{\text{use the generating function (2.2)}\} \\
&= \int_0^1 \frac{\mathrm{Li}_{2q+1}(-x^2)}{1+x^2}\mathrm{d}x.
\end{aligned}$$

This integral is given in (3.76). For a different proof, set $q = 0$ in (4.38).

Examples:

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{2n+1} = G - \frac{\pi}{8}\ln(2);$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(3)}}{2n+1} = 4\beta(4) + G\zeta(2) - \frac{3\pi}{2}\zeta(3);$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(5)}}{2n+1} = 16\beta(6) + 4\zeta(2)\beta(4) + \frac{7}{4}G\zeta(4) - \frac{15\pi}{2}\zeta(5);$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(7)}}{2n+1} = 64\beta(8) + 16\zeta(2)\beta(6) + 7\zeta(4)\beta(4) + \frac{31}{16}G\zeta(6) - \frac{63\pi}{2}\zeta(7).$$

4.1.19 $\sum_{n=0}^{\infty}\frac{(-1)^n H_{2n+1}}{(2n+1)^{2q+1}}$

Let $q \in \mathbb{Z}_{\geq 0}$. Then the following identity holds:

$$\sum_{n=0}^{\infty}\frac{(-1)^n H_{2n+1}}{(2n+1)^{2q+1}} = (q+1)\beta(2q+2) - \frac{\pi}{2^{2q+3}}\,\eta(2q+1) - \sum_{k=1}^{q}\zeta(2k+1)\beta(2q-2k+1). \tag{4.36}$$

Proof. By expanding $\frac{1}{1+x^2}$ in Taylor series, we have

$$\begin{aligned}
\int_0^1 \frac{\operatorname{Li}_{2q+1}(x)}{1+x^2}\mathrm{d}x &= \int_0^1 \operatorname{Li}_{2q+1}(x)\left(\sum_{n=0}^{\infty}\left(-x^2\right)^n\right)\mathrm{d}x\\
&= \sum_{n=0}^{\infty}(-1)^n\int_0^1 x^{2n}\operatorname{Li}_{2q+1}(x)\mathrm{d}x\\
&\qquad\{\text{make use of the result (3.65)}\}\\
&= \sum_{n=0}^{\infty}(-1)^n\left(\frac{H_{2n+1}}{(2n+1)^{2q+1}} - \sum_{k=1}^{2q}(-1)^k\frac{\zeta(2q-k+2)}{(2n+1)^k}\right)\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n H_{2n+1}}{(2n+1)^{2q+1}} - \sum_{k=1}^{2q}(-1)^k\zeta(2q-k+2)\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^k}\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n H_{2n+1}}{(2n+1)^{2q+1}} - \sum_{k=1}^{2q}(-1)^k\zeta(2q-k+2)\beta(k).
\end{aligned}$$

Substitute the result of the integral given in (3.75), we get

$$\sum_{n=0}^{\infty}\frac{(-1)^n H_{2n+1}}{(2n+1)^{2q+1}} = \frac{2q+1}{2}\beta(2q+2) - \frac{\pi}{2^{2q+3}}\,\eta(2q+1) - \sum_{k=0}^{q}\zeta(2q-2k)\beta(2k+2) + \sum_{k=1}^{2q}(-1)^k\zeta(2q-k+2)\beta(k).$$

For the first sum, separate the first term then reverse the order of the terms,

$$\sum_{k=0}^{q}\zeta(2q-2k)\beta(2k+2) = -\frac{1}{2}\beta(2q+2) + \sum_{k=1}^{q}\zeta(2q-2k+2)\beta(2k).$$

For the second sum, first reverse the order of the terms then use $\sum_{k=1}^{2q} a_k = \sum_{k=1}^{q} a_{2k-1} + \sum_{k=1}^{q} a_{2k}$:

$$\begin{aligned}\sum_{k=1}^{2q} (-1)^k \zeta(2q-k+2)\beta(k) &= \sum_{k=1}^{2q} (-1)^{k-1} \zeta(k+1)\beta(2q-k+1)\\ &= \sum_{k=1}^{q} \zeta(2k)\beta(2q-2k+2) - \sum_{k=1}^{q} \zeta(2k+1)\beta(2q-2k+1).\end{aligned}$$

Plug in these two sums to complete the proof.

It's worth to mention that by using (1.13), we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n H_{2n+1}}{(2n+1)^{2q+1}} = \Im \sum_{n=1}^{\infty} \frac{H_n}{n^{2q+1}}.$$

Examples:

$$\sum_{n=0}^{\infty} \frac{(-1)^n H_{2n+1}}{2n+1} = G - \frac{\pi}{8}\ln(2);$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n H_{2n+1}}{(2n+1)^3} = 2\beta(4) - \frac{35\pi}{128}\zeta(3);$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n H_{2n+1}}{(2n+1)^5} = 3\beta(6) - \frac{\pi^3}{32}\zeta(3) - \frac{527\pi}{2048}\zeta(5);$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n H_{2n+1}}{(2n+1)^7} = 4\beta(8) - \beta(5)\zeta(3) - \frac{\pi^3}{32}\zeta(5) - \frac{8255\pi}{32768}\zeta(7).$$

4.1.20 $\boldsymbol{\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{(2n+1)^{2q+1}}}$

Let $q \in \mathbb{Z}_{\geq 0}$. Then the following identity holds:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{(2n+1)^{2q+1}} &= (2q+1)\beta(2q+2) - \frac{\ln(2)|E_{2q}|}{(2q)!}\left(\frac{\pi}{2}\right)^{2q+1}\\ &-\frac{1}{2}\left(\frac{\pi}{2}\right)^{2q+1} \sum_{k=1}^{q} \frac{|E_{2q-2k}|}{(2q-2k)!\pi^{2k}}(2^{2k+1}-1)\zeta(2k+1). \qquad (4.37)\end{aligned}$$

Proof.

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{(2n+1)^{2q+1}} &= \sum_{n=1}^{\infty}(-1)^n H_n\left(\frac{1}{(2q)!}\int_0^1 x^{2n}\ln^{2q}(x)\mathrm{d}x\right)\\ &= \frac{1}{(2a)!}\int_0^1 \ln^{2q}(x)\left(\sum_{n=1}^{\infty}H_n(-x^2)^n\right)\mathrm{d}x\\ &\qquad\{\text{use the generating function (2.4)}\}\\ &= \frac{1}{(2q)!}\int_0^1 \ln^{2q}(x)\left(-\frac{\ln(1+x^2)}{1+x^2}\right)\mathrm{d}x\end{aligned}$$

and this integral is given in (3.35).

Examples:

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{2n+1} = G+\frac{\pi}{4};$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{(2n+1)^3} = 3\beta(4)-\frac{7\pi}{16}\zeta(3)-\frac{\pi^3}{16}\ln(2);$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{(2n+1)^5} = 5\beta(6)-\frac{31\pi}{64}\zeta(5)-\frac{7\pi^3}{128}\zeta(3)-\frac{5\pi^5}{768}\ln(2);$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{(2n+1)^7} = 7\beta(8)-\frac{127\pi}{256}\zeta(7)-\frac{31\pi^3}{512}\zeta(5)-\frac{35\pi^5}{6144}\zeta(3)-\frac{61\pi^7}{92160}\ln(2).$$

4.1.21 $\sum_{n=1}^{\infty}\frac{(-1)^n H_n^{(2p+1)}}{(2n+1)^{2q+1}}$

Let $p\in\mathbb{Z}^+$ and $q\in\mathbb{Z}_{\geq 0}$. Then the following identity holds:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n H_n^{(2p+1)}}{(2n+1)^{2q+1}} &= \frac{|E_{2q}|}{2(2q)!}\left(\frac{\pi}{2}\right)^{2q+1}\zeta(2p+1)\\ -\frac{\left(\frac{\pi}{2}\right)^{2q+1}}{2(2p)!(2q)!}&\sum_{j=0}^{q}\binom{2q}{2j}\frac{(2p+2j)!}{\pi^{2j}}|E_{2q-2j}|(2^{2j+2p+1}-1)\zeta(2j+2p+1)\\ &+\frac{2}{(2q)!}\sum_{k=0}^{p}\frac{(2k+2q+1)!}{(2k+1)!}2^{2k}\eta(2p-2k)\beta(2k+2q+2).\end{aligned}\tag{4.38}$$

Proof.

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n H_n^{(2p+1)}}{(2n+1)^{2q+1}} &= \sum_{n=1}^{\infty}(-1)^n H_n^{(2p+1)}\left(\frac{1}{(2q)!}\int_0^1 x^{2n}\ln^{2q}(x)\mathrm{d}x\right)\\ &= \frac{1}{(2q)!}\int_0^1 \ln^{2q}(x)\left(\sum_{n=1}^{\infty}H_n^{(2p+1)}(-x^2)^n\right)\mathrm{d}x\\ &= \frac{1}{(2q)!}\int_0^1 \ln^{2q}(x)\left(\frac{\mathrm{Li}_{2p+1}(-x^2)}{1+x^2}\right)\mathrm{d}x,\end{aligned}$$

which is found in (3.81).

Examples:

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n^{(3)}}{(2n+1)^5} = 140\beta(8)+5\zeta(2)\beta(6)-\frac{1905}{64}\pi\zeta(7)-\frac{93}{64}\pi^3\zeta(5)-\frac{5\pi^5}{256}\zeta(3);$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n^{(5)}}{(2n+1)^3} = 336\beta(8)+40\zeta(2)\beta(6)+\frac{21}{4}\zeta(4)\beta(4)-\frac{1905\pi}{16}\zeta(7) -\frac{15\pi^3}{16}\zeta(5);$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n^{(5)}}{(2n+1)^7} = 7392\beta(12)+336\zeta(2)\beta(10)+\frac{49}{4}\zeta(4)\beta(8)-\frac{214935}{128}\pi\zeta(11) -\frac{17885}{256}\pi^3\zeta(9)-\frac{3175}{2048}\pi^5\zeta(7)-\frac{61\pi^7}{6144}\zeta(5);$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n^{(7)}}{(2n+1)^5} = 21120\beta(12)+2016\zeta(2)\beta(10)+245\zeta(4)\beta(8)+\frac{155}{16}\zeta(6)\beta(6) -\frac{214935}{32}\pi\zeta(11)-\frac{3577}{32}\pi^3\zeta(9)-\frac{105}{256}\pi^5\zeta(7).$$

4.1.22 $\sum_{n=1}^{\infty}\frac{(-1)^n H_n^{(2p)}}{(2n+1)^{2q}}$

Let $q,\ p\in\mathbb{Z}^+$. Then the following identity holds:

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n^{(2p)}}{(2n+1)^{2q}} = \frac{\left(\frac{\pi}{2}\right)^{2q-1}}{4(2p-1)!(2q-1)!}* \sum_{j=0}^{q-1}\binom{2q-1}{2j+1}\frac{(2p+2j)!}{\pi^{2j}}\left|E_{2q-2j-2}\right|\left(2^{2j+2p+1}-1\right)\zeta(2j+2p+1)$$

(continued)

$$-\frac{1}{(2q-1)!}\sum_{k=0}^{p}\frac{(2k+2q-1)!}{(2k)!}2^{2k}\eta(2p-2k)\beta(2k+2q). \tag{4.39}$$

Proof. Applying the previous proof, we get

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n H_n^{(2p)}}{(2n+1)^{2q}} &= \sum_{n=1}^{\infty}(-1)^n H_n^{(2p)}\left(\frac{-1}{(2q-1)!}\int_0^1 x^{2n}\ln^{2q-1}(x)\mathrm{d}x\right)\\ &= \frac{-1}{(2q-1)!}\int_0^1 \ln^{2q-1}(x)\left(\sum_{n=1}^{\infty}H_n^{(2p)}(-x^2)^n\right)\mathrm{d}x\\ &= \frac{-1}{(2q-1)!}\int_0^1 \ln^{2q-1}(x)\left(\frac{\operatorname{Li}_{2p}(-x^2)}{1+x^2}\right)\mathrm{d}x\end{aligned}$$

and this integral is given in (3.82).

Examples:

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n^{(2)}}{(2n+1)^4} = \frac{7\pi^3}{32}\zeta(3)+\frac{31\pi}{8}\zeta(5)-\frac{1}{2}\zeta(2)\beta(4)-20\beta(6);$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n^{(4)}}{(2n+1)^2} = \frac{31\pi}{2}\zeta(5)-\frac{7}{8}\zeta(4)\beta(2)-6\zeta(2)\beta(4)-40\beta(6);$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n^{(4)}}{(2n+1)^6} = \frac{155}{768}\pi^5\zeta(5)+\frac{635}{64}\pi^3\zeta(7)+\frac{3577}{16}\pi\zeta(9)-\frac{7}{8}\zeta(4)\beta(6)$$
$$-42\zeta(2)\beta(8)-1008\beta(10);$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n^{(6)}}{(2n+1)^4} = \frac{381}{32}\pi^3\zeta(7)+\frac{3577}{4}\pi\zeta(9)-\frac{31}{32}\zeta(6)\beta(4)-35\zeta(4)\beta(6)$$
$$-280\zeta(2)\beta(8)-2688\beta(10).$$

4.2 Non–Alternating Harmonic Series

4.2.1 $\sum_{n=1}^{\infty}\frac{H_n}{n^2}$

Show that

$$\sum_{n=1}^{\infty}\frac{H_n}{n^2} = 2\zeta(3). \tag{4.40}$$

Solution. Set $s = 0$ and $q = 2$ in (4.3), we get

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} = \int_0^1 \frac{\ln(x)\ln(1-x)}{x(1-x)} \mathrm{d}x$$
$$\{\text{set } q = 1 \text{ in (3.83)}\}$$
$$= 2\int_0^1 \frac{\ln(x)\ln(1-x)}{x} \mathrm{d}x.$$

This integral is given in (3.9).
For a different approach, set $x = 1$ in (2.8).

4.2.2 $\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2}$

Show that

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2} = \frac{17}{4}\zeta(4). \tag{4.41}$$

Solution. Put $x = 1$ in (2.69),

$$\operatorname{Li}_2^2(1) = 4\sum_{n=1}^{\infty} \frac{H_n}{n^3} + 2\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2} - 6\operatorname{Li}_4(1).$$

The first sum is given in (4.4) and remember that $\operatorname{Li}_a(1) = \zeta(a)$.
Check (2.77) for a different solution.

4.2.3 $\sum_{n=1}^{\infty} \frac{H_n^2}{n^2}$

Show that

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^2} = \frac{17}{4}\zeta(4). \tag{4.42}$$

Solution (i).

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^2} = \sum_{n=1}^{\infty} \frac{H_n}{n}\left(\frac{H_n}{n}\right)$$
$$\{\text{recall the integral (1.124)}\}$$

$$= \sum_{n=1}^{\infty} \frac{H_n}{n} \left(- \int_0^1 x^{n-1} \ln(1-x) \mathrm{d}x \right)$$

$$= - \int_0^1 \frac{\ln(1-x)}{x} \left(\sum_{n=1}^{\infty} \frac{H_n}{n} x^n \right) \mathrm{d}x$$

{recall the generating function (2.6)}

$$= - \int_0^1 \frac{\ln(1-x)}{x} \left(\mathrm{Li}_2(x) + \frac{1}{2} \ln^2(1-x) \right) \mathrm{d}x$$

$$= \frac{1}{2} \mathrm{Li}_2^2(x)\Big|_0^1 - \frac{1}{2} \int_0^1 \frac{\ln^3(1-x)}{x} \mathrm{d}x$$

$$\overset{1-x=y}{=} \frac{5}{4} \zeta(4) - \frac{1}{2} \int_0^1 \frac{\ln^3(y)}{1-y} \mathrm{d}y.$$

This integral is given in (3.8) and the solution is complete.

Solution (ii).

$$\sum_{n=1}^{\infty} \frac{H_n^2 + H_n^{(2)}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{H_n^2 + H_n^{(2)}}{n} \right)$$

{recall the integral (2.62)}

$$= \sum_{n=1}^{\infty} \frac{1}{n} \left(\int_0^1 x^{n-1} \ln^2(1-x) \mathrm{d}x \right)$$

$$= \int_0^1 \frac{\ln^2(1-x)}{x} \left(\sum_{n=1}^{\infty} \frac{x^n}{n} \right) \mathrm{d}x$$

$$= \int_0^1 \frac{\ln^2(1-x)}{x} \left(- \ln(1-x) \right) \mathrm{d}x$$

$$\overset{1-x=y}{=} - \int_0^1 \frac{\ln^3(y)}{1-y} \mathrm{d}y.$$

Group the values (3.8) and (4.41) to finish the solution.

Solution (iii). Substitute $\frac{1}{n^2} = - \int_0^1 x^{n-1} \ln(x) \mathrm{d}x$ to get

$$\sum_{n=1}^{\infty} \frac{H_n^2 - H_n^{(2)}}{n^2} = \sum_{n=1}^{\infty} (H_n^2 - H_n^{(2)}) \left(- \int_0^1 x^{n-1} \ln(x) \mathrm{d}x \right)$$

$$= - \int_0^1 \frac{\ln(x)}{x} \left(\sum_{n=1}^{\infty} (H_n^2 - H_n^{(2)}) x^n \right) \mathrm{d}x$$

{recall the generating function (2.11)}

$$= -\int_0^1 \frac{\ln(x)}{x}\left(\frac{\ln^2(1-x)}{1-x}\right)\mathrm{d}x$$

$$\overset{1-x=y}{=} -\int_0^1 \frac{\ln^2(y)\ln(1-y)}{y(1-y)}\mathrm{d}y.$$

Collect the results (3.44) and (4.41) to complete the solution.
For different methods of evaluating (4.41) and (4.42), check [25].

4.2.4 $\sum_{n=1}^{\infty} \frac{H_n H_{2n}}{n^2}$

Show that

$$\sum_{n=1}^{\infty} \frac{H_n H_{2n}}{n^2} = 4\operatorname{Li}_4\left(\frac{1}{2}\right) + \frac{13}{8}\zeta(4) + \frac{7}{2}\ln(2)\zeta(3) - \ln^2(2)\zeta(2) + \frac{1}{6}\ln^4(2). \tag{4.43}$$

Solution.

$$\sum_{n=1}^{\infty} \frac{H_n H_{2n} - 2H_{2n}^2}{n^2} = \sum_{n=1}^{\infty} \frac{H_n - 2H_{2n}}{n}\left(\frac{H_{2n}}{n}\right)$$

$$\{\text{replace } n \text{ with } 2n \text{ in (1.124)}\}$$

$$= \sum_{n=1}^{\infty} \frac{H_n - 2H_{2n}}{n}\left(-2\int_0^1 x^{2n-1}\ln(1-x)\mathrm{d}x\right)$$

$$= \int_0^1 \frac{\ln(1-x)}{x}\left(2\sum_{n=1}^{\infty} \frac{2H_{2n} - H_n}{n} x^{2n}\right)\mathrm{d}x$$

$$\{\text{recall the generating function (2.40)}\}$$

$$= \int_0^1 \frac{\ln(1-x)}{x}\left(\ln^2\left(\frac{1-x}{1+x}\right)\right)\mathrm{d}x$$

$$\left\{\text{use } a(a-b)^2 = \frac{1}{6}(a+b)^3 + \frac{1}{6}(a-b)^3 + \frac{2}{3}a^3 - 2a^2 b\right\}$$

$$\{\text{with } a = \ln(1-x) \text{ and } b = \ln(1+x)\}$$

$$= \underbrace{\frac{1}{6}\int_0^1 \frac{\ln^3(1-x^2)}{x}\mathrm{d}x}_{1-x^2\to x} + \frac{1}{6}\int_0^1 \frac{\ln^3\left(\frac{1-x}{1+x}\right)}{x}\mathrm{d}x + \underbrace{\frac{2}{3}\int_0^1 \frac{\ln^3(1-x)}{x}\mathrm{d}x}_{1-x\to x}$$

$$-2\int_0^1 \frac{\ln^2(1-x)\ln(1+x)}{x}\mathrm{d}x$$

$$= \frac{3}{4}\int_0^1 \frac{\ln^3(x)}{1-x}\mathrm{d}x + \frac{1}{6}\int_0^1 \frac{\ln^3\left(\frac{1-x}{1+x}\right)}{x}\mathrm{d}x - 2\int_0^1 \frac{\ln^2(1-x)\ln(1+x)}{x}\mathrm{d}x$$

$$\{\text{collect the results (3.8), (3.15), and (3.116)}\}$$

$$= -\frac{2}{3}\operatorname{Li}_4\left(\frac{1}{2}\right) - \frac{41}{48}\zeta(4) - \frac{7}{12}\ln(2)\zeta(3) + \frac{1}{6}\ln^2(2)\zeta(2) - \frac{1}{36}\ln^4(2).$$

For the remaining sum, make use of (1.5) to have

$$\sum_{n=1}^{\infty}\frac{H_{2n}^2}{n^2} = 4\sum_{n=1}^{\infty}\frac{H_{2n}^2}{(2n)^2} = 2\sum_{n=1}^{\infty}\frac{H_n^2}{n^2} + 2\sum_{n=1}^{\infty}\frac{(-1)^n H_n^2}{n^2}$$

$$\{\text{substitute the results (4.42) and (4.88)}\}$$

$$= 4\operatorname{Li}_4\left(\frac{1}{2}\right) + \frac{27}{8}\zeta(4) + \frac{7}{2}\ln(2)\zeta(3) - \ln^2(2)\zeta(2) + \frac{1}{6}\ln^4(2).$$

4.2.5 $\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3}$

Show that

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3} = 3\zeta(2)\zeta(3) - \frac{9}{2}\zeta(5). \tag{4.44}$$

Solution. Setting $x = 1$ in (2.71) gives

$$3\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3} + \sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^2} = \zeta(2)\zeta(3) + 10\zeta(5) - 6\sum_{n=1}^{\infty}\frac{H_n}{n^4}$$

$$\{\text{plug in the result (4.4)}\}$$

$$= 7\zeta(2)\zeta(3) - 8\zeta(5). \tag{4.45}$$

Now let's set $p = 2$ and $q = 3$ in (2.76) to have

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3} + \sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^2} = \zeta(2)\zeta(3) + \zeta(5). \tag{4.46}$$

Take the difference of these two relations to end the solution.

Also see (4.21) for another method.

4.2.6 $\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^2}$

Show that

$$\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^2} = \frac{11}{2}\zeta(5) - 2\zeta(2)\zeta(3). \tag{4.47}$$

Solution. Combine (4.45) and (4.46) after multiplying the latter by -3.

4.2.7 $\sum_{n=1}^{\infty} \frac{H_n^2}{n^3}$

Show that

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^3} = \frac{7}{2}\zeta(5) - \zeta(2)\zeta(3). \tag{4.48}$$

Solution (i). Substitute $\frac{1}{n^3} = \frac{1}{2}\int_0^1 x^{n-1}\ln^2(x)\mathrm{d}x$ to have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{H_n^2 - H_n^{(2)}}{n^3} &= \sum_{n=1}^{\infty} (H_n^2 - H_n^{(2)}) \left(\frac{1}{2}\int_0^1 x^{n-1}\ln^2(x)\mathrm{d}x\right) \\
&= \frac{1}{2}\int_0^1 \frac{\ln^2(x)}{x}\left(\sum_{n=1}^{\infty}(H_n^2 - H_n^{(2)})x^n\right)\mathrm{d}x \\
&\qquad \{\text{recall the generating function (2.11)}\} \\
&= \frac{1}{2}\int_0^1 \frac{\ln^2(x)}{x}\left(\frac{\ln^2(1-x)}{1-x}\right)\mathrm{d}x \\
&\qquad \{\text{set } q = 2 \text{ in (3.83)}\} \\
&= \int_0^1 \frac{\ln^2(1-x)\ln^2(x)}{x} \\
&\overset{\text{IBP}}{=} \frac{2}{3}\int_0^1 \frac{\ln^3(x)\ln(1-x)}{1-x}\mathrm{d}x.
\end{aligned}$$

The solution completes on grouping (4.44) and (3.41).
Solution (ii).

$$\sum_{n=1}^{\infty} \frac{H_{n-1}H_n}{n^3} = \sum_{n=1}^{\infty} \frac{H_{n-1}}{n}\left(\frac{H_n}{n^2}\right)$$
$$\{\text{recall the integral (3.66)}\}$$

$$= \sum_{n=1}^{\infty} \frac{H_{n-1}}{n} \left(\int_0^1 \frac{\zeta(2) - \operatorname{Li}_2(x)}{x} x^n \mathrm{d}x \right)$$

$$= \int_0^1 \frac{\zeta(2) - \operatorname{Li}_2(x)}{x} \left(\sum_{n=1}^{\infty} \frac{H_{n-1}}{n} x^n \right) \mathrm{d}x$$

$$\{\text{recall the generating function (2.7)}\}$$

$$= \int_0^1 \frac{\zeta(2) - \operatorname{Li}_2(x)}{x} \left(\frac{1}{2} \ln^2(1-x) \right) \mathrm{d}x$$

$$\{\text{set } 1 - x = y \text{ in the first integral}\}$$

$$= \frac{1}{2}\zeta(2) \int_0^1 \frac{\ln^2(y)}{1-y} \mathrm{d}y - \frac{1}{2} \int_0^1 \frac{\ln^2(1-x) \operatorname{Li}_2(x)}{x} \mathrm{d}x.$$

By writing

$$\sum_{n=1}^{\infty} \frac{H_{n-1} H_n}{n^3} = \sum_{n=1}^{\infty} \frac{\left(H_n - \frac{1}{n}\right) H_n}{n^3} = \sum_{n=1}^{\infty} \frac{H_n^2}{n^3} - \sum_{n=1}^{\infty} \frac{H_n}{n^4},$$

we get

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^3} = \frac{1}{2}\zeta(2) \int_0^1 \frac{\ln^2(y)}{1-y} \mathrm{d}y - \frac{1}{2} \int_0^1 \frac{\ln^2(1-x) \operatorname{Li}_2(x)}{x} \mathrm{d}x + \sum_{n=1}^{\infty} \frac{H_n}{n^4}.$$

These terms are given in (3.8), (3.69), and (4.4) respectively.
Also check (4.26) for another solution.

4.2.8 $\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^2}$

Show that

$$\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^2} = \zeta(5) + \zeta(2)\zeta(3). \tag{4.49}$$

Solution (i).

$$\sum_{n=1}^{\infty} \frac{H_n^3 + 3H_n H_n^{(2)} + 2H_n^{(3)}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{H_n^3 + 3H_n H_n^{(2)} + 2H_n^{(3)}}{n} \right)$$

$$\{\text{recall the integral (2.62)}\}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \left(- \int_0^1 x^{n-1} \ln^3(1-x) \mathrm{d}x \right)$$

$$
\begin{aligned}
&= -\int_0^1 \frac{\ln^3(1-x)}{x}\left(\sum_{n=1}^{\infty}\frac{x^n}{n}\right)\mathrm{d}x \\
&= -\int_0^1 \frac{\ln^3(1-x)}{x}\left(-\ln(1-x)\right)\mathrm{d}x \\
&\overset{1-x=y}{=} \int_0^1 \frac{\ln^4(y)}{1-y}\mathrm{d}y \\
&\quad\{\text{this integral is given in (3.8)}\} \\
&= 24\zeta(5). \qquad (4.50)
\end{aligned}
$$

Now write $\frac{1}{n^2} = -\int_0^1 x^{n-1}\ln(x)\mathrm{d}x$ to get

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\frac{H_n^3 - 3H_nH_n^{(2)} + 2H_n^{(3)}}{n^2} \\
&= \sum_{n=1}^{\infty}\left(H_n^3 - 3H_nH_n^{(2)} + 2H_n^{(3)}\right)\left(-\int_0^1 x^{n-1}\ln(x)\mathrm{d}x\right) \\
&= -\int_0^1 \frac{\ln(x)}{x}\left(\sum_{n=1}^{\infty}\left(H_n^3 - 3H_nH_n^{(2)} + 2H_n^{(3)}\right)x^n\right)\mathrm{d}x \\
&\quad\{\text{recall the generating function (2.22)}\} \\
&= -\int_0^1 \frac{\ln(x)}{x}\left(-\frac{\ln^3(1-x)}{1-x}\right)\mathrm{d}x \\
&\overset{1-x=y}{=} \int_0^1 \frac{\ln^3(y)\ln(1-y)}{y(1-y)}\mathrm{d}y \\
&\quad\{\text{this integral is calculated in (3.44)}\} \\
&= 18\zeta(5) - 6\zeta(2)\zeta(3). \qquad (4.51)
\end{aligned}
$$

Take the difference of (4.50) and (4.51) to finish the solution.

Solution (ii). Divide both sides of (2.7):

$$\sum_{n=1}^{\infty}\frac{H_{n-1}}{n}y^n = \frac{1}{2}\ln^2(1-y)$$

by y then integrate from $y = 0$ to x using $\int_0^x y^{n-1}\mathrm{d}y = \frac{x^n}{n}$,

$$\sum_{n=1}^{\infty}\frac{H_{n-1}}{n^2}x^n = \frac{1}{2}\int_0^x \frac{\ln^2(1-y)}{y}\mathrm{d}y. \qquad (4.52)$$

Next, we consider the following double sum:

$$\begin{aligned}
\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{H_{n-1}}{n^2(n+k)^2} &= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{H_{n-1}}{n^2}\left(-\int_0^1 x^{n+k-1}\ln(x)\mathrm{d}x\right)\\
&= -\int_0^1 \ln(x)\left(\sum_{k=1}^{\infty}x^{k-1}\right)\left(\sum_{n=1}^{\infty}\frac{H_{n-1}}{n^2}x^n\right)\mathrm{d}x\\
&\qquad \{\text{the second sum is given in (4.52)}\}\\
&= -\int_0^1 \ln(x)\left(\frac{1}{1-x}\right)\left(\frac{1}{2}\int_0^x \frac{\ln^2(1-y)}{y}\mathrm{d}y\right)\mathrm{d}x\\
&\qquad \{\text{change the order of integration}\}\\
&= -\frac{1}{2}\int_0^1 \frac{\ln^2(1-y)}{y}\left(\int_y^1 \frac{\ln(x)}{1-x}\mathrm{d}x\right)\mathrm{d}y\\
&= -\frac{1}{2}\int_0^1 \frac{\ln^2(1-y)}{y}\left(\operatorname{Li}_2(1-x)\Big|_y^1\right)\mathrm{d}y\\
&= -\frac{1}{2}\int_0^1 \frac{\ln^2(1-y)}{y}\left(-\operatorname{Li}_2(1-y)\right)\mathrm{d}y\\
&\overset{1-x=y}{=} \frac{1}{2}\int_0^1 \frac{\ln^2(y)\operatorname{Li}_2(y)}{1-y}\mathrm{d}y\\
&\qquad \left\{\text{expand } \frac{\operatorname{Li}_2(y)}{1-y} \text{ in series given in (2.3)}\right\}\\
&= \frac{1}{2}\sum_{n=1}^{\infty}H_{n-1}^{(2)}\left(\int_0^1 y^{n-1}\ln^2(x)\mathrm{d}y\right)\\
&= \frac{1}{2}\sum_{n=1}^{\infty}\left(H_n^{(2)}-\frac{1}{n^2}\right)\left(\frac{2}{n^3}\right)\\
&= \sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3}-\zeta(5). \qquad (4.53)
\end{aligned}$$

On the other hand, set $a = 2$ in (1.131) to get

$$\sum_{k=1}^{\infty}\frac{1}{(n+k)^2} = \zeta(2) - H_n^{(2)}.$$

Multiply both sides by $\frac{H_{n-1}}{n}$ then take the summation over $n \geq 1$,

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{H_{n-1}}{n^2(n+k)^2} = \sum_{n=1}^{\infty}\frac{H_{n-1}}{n^2}\left(\zeta(2)-H_n^{(2)}\right)$$

$$= \zeta(2)\sum_{n=1}^{\infty}\frac{H_{n-1}}{n^2} - \sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^2} + \sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3}. \tag{4.54}$$

Combine (4.54) and (4.53),

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^2} &= \zeta(5) + \zeta(2)\sum_{n=1}^{\infty}\frac{H_{n-1}}{n^2} \\ &\qquad \{\text{set } x = 1 \text{ in } (4.52)\} \\ &= \zeta(5) + \frac{1}{2}\zeta(2)\int_0^1 \frac{\ln^2(1-y)}{y}\mathrm{d}y \\ &= \zeta(5) + \frac{1}{2}\zeta(2)\int_0^1 \frac{\ln^2(x)}{1-x}\mathrm{d}x.\end{aligned}$$

This integral is calculated in (3.8).

4.2.9 $\sum_{n=1}^{\infty}\frac{H_n^3}{n^2}$

Show that

$$\sum_{n=1}^{\infty}\frac{H_n^3}{n^2} = 10\zeta(5) + \zeta(2)\zeta(3). \tag{4.55}$$

Solution. Combining (4.50) and (4.51) then dividing by 2 yields,

$$\sum_{n=1}^{\infty}\frac{H_n^3}{n^2} = 21\zeta(5) - 3\zeta(2)\zeta(3) - 2\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^2}.$$

This sum is evaluated in (4.111).

For a different approach, see [32, pp. 401–402].

4.2.10 $\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4}$

Show that

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4} = \zeta^2(3) - \frac{1}{3}\zeta(6). \tag{4.56}$$

Solution. Set $x = 1$ in (2.72),

$$\zeta^2(3) = 12\sum_{n=1}^{\infty}\frac{H_n}{n^6} + 2\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^3} + 6\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4} - 20\zeta(6).$$

The first two sums are calculated in (4.4) and (2.77) respectively.

4.2.11 $\sum_{n=1}^{\infty}\frac{H_n^{(4)}}{n^2}$

Show that

$$\sum_{n=1}^{\infty}\frac{H_n^{(4)}}{n^2} = \frac{37}{12}\zeta(6) - \zeta^2(3). \tag{4.57}$$

Solution. Putting $p = 2$ and $q = 4$ in (2.76) gives

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4} + \sum_{n=1}^{\infty}\frac{H_n^{(4)}}{n^2} = \zeta(2)\zeta(4) + \zeta(6).$$

The first sum is given in (4.56) and note that $\zeta(2)\zeta(4) = \frac{7}{4}\zeta(6)$.

4.2.12 $\sum_{n=1}^{\infty}\frac{H_n^2}{n^4}$

Show that

$$\sum_{n=1}^{\infty}\frac{H_n^2}{n^4} = \frac{97}{24}\zeta(6) - 2\zeta^2(3). \tag{4.58}$$

Solution.

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{H_n^2 + H_n^{(2)}}{n^4} &= \sum_{n=1}^{\infty}\frac{1}{n^3}\left(\frac{H_n^2 + H_n^{(2)}}{n}\right) \\
&\quad \{\text{recall the integral (2.62)}\} \\
&= \sum_{n=1}^{\infty}\frac{1}{n^3}\left(\int_0^1 x^{n-1}\ln^2(1-x)\mathrm{d}x\right) \\
&= \int_0^1 \frac{\ln^2(1-x)}{x}\left(\sum_{n=1}^{\infty}\frac{x^n}{n^3}\right)\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \frac{\ln^2(1-x)\,\mathrm{Li}_3(x)}{x}\mathrm{d}x \\
&\qquad \{\text{expand } \ln^2(1-x) \text{ in series given in (2.7)}\} \\
&= 2\sum_{n=1}^{\infty} \frac{H_{n-1}}{n}\left(\int_0^1 x^{n-1}\,\mathrm{Li}_3(x)\mathrm{d}x\right) \\
&\qquad \{\text{recall the result of the integral from (3.67)}\} \\
&= 2\sum_{n=1}^{\infty}\left(\frac{H_n}{n}-\frac{1}{n^2}\right)\left(\frac{\zeta(3)}{n}-\frac{\zeta(2)}{n^2}+\frac{H_n}{n^3}\right) \\
&= 2\zeta(3)\sum_{n=1}^{\infty}\frac{H_n}{n^2}-2\zeta(2)\sum_{n=1}^{\infty}\frac{H_n}{n^3}-2\sum_{n=1}^{\infty}\frac{H_n}{n^5}+2\sum_{n=1}^{\infty}\frac{H_n^2}{n^4}.
\end{aligned}
$$

Rearrange the terms, we arrive at

$$
\sum_{n=1}^{\infty}\frac{H_n^2}{n^4} = -2\zeta(3)\sum_{n=1}^{\infty}\frac{H_n}{n^2}+2\zeta(2)\sum_{n=1}^{\infty}\frac{H_n}{n^3}+2\sum_{n=1}^{\infty}\frac{H_n}{n^5}-\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4}.
$$

The first three sums are given in (4.4) and the last is given in (4.56).
The two series in (4.56) and (4.58) may be found evaluated differently in [25].

4.2.13 $\sum_{n=1}^{\infty} \frac{\left(H_n^{(2)}\right)^2}{n^2}$

Show that

$$
\sum_{n=1}^{\infty}\frac{\left(H_n^{(2)}\right)^2}{n^2} = \frac{35}{24}\zeta(6)-\zeta^2(3). \tag{4.59}
$$

Solution. Substitute (4.56) in (2.6.3).

4.2.14 $\sum_{n=1}^{\infty} \frac{H_n H_n^{(3)}}{n^2}$

Show that

$$
\sum_{n=1}^{\infty}\frac{H_n H_n^{(3)}}{n^2} = \frac{227}{48}\zeta(6)-\frac{3}{2}\zeta^2(3). \tag{4.60}
$$

Solution.

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{H_nH_n^{(3)}}{n^2}&=\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n}\left(\frac{H_n}{n}\right)\\
&=\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n}\left(-\int_0^1 x^{n-1}\ln(1-x)\mathrm{d}x\right)\\
&=-\int_0^1\frac{\ln(1-x)}{x}\left(\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n}x^n\right)\mathrm{d}x\\
&\qquad\{\text{recall the generating function (2.17)}\}\\
&=-\int_0^1\frac{\ln(1-x)}{x}\left(\operatorname{Li}_4(x)-\ln(1-x)\operatorname{Li}_3(x)-\frac{1}{2}\operatorname{Li}_2^2(x)\right)\mathrm{d}x\\
&=\int_0^1\frac{\ln^2(1-x)\operatorname{Li}_3(x)}{x}\mathrm{d}x-\int_0^1\frac{\ln(1-x)\operatorname{Li}_4(x)}{x}\mathrm{d}x\\
&\quad+\frac{1}{2}\int_0^1\frac{\ln(1-x)\operatorname{Li}_2^2(x)}{x}\mathrm{d}x\\
&\qquad\{\text{expand }\operatorname{Li}_3(x)\text{ and }\operatorname{Li}_4(x)\text{ in series}\}\\
&=\sum_{n=1}^{\infty}\frac{1}{n^3}\int_0^1 x^{n-1}\ln^2(1-x)\mathrm{d}x-\sum_{n=1}^{\infty}\frac{1}{n^4}\int_0^1 x^{n-1}\ln(1-x)\mathrm{d}x-\frac{1}{6}\operatorname{Li}_2^3(x)\Big|_0^1\\
&\qquad\{\text{make use of (2.62) for the first integral}\}\\
&=\sum_{n=1}^{\infty}\frac{1}{n^3}\left(\frac{H_n^2}{n}+\frac{H_n^{(2)}}{n}\right)-\sum_{n=1}^{\infty}\frac{1}{n^4}\left(-\frac{H_n}{n}\right)-\frac{1}{6}\zeta^3(2)\\
&=\sum_{n=1}^{\infty}\frac{H_n}{n^5}+\sum_{n=1}^{\infty}\frac{H_n^2}{n^4}+\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4}-\frac{35}{48}\zeta(6).
\end{aligned}$$

These sums are found in (4.4), (4.58), and (4.56). Note that $\zeta^3(2)=\frac{35}{8}\zeta(6)$. Also check [32, p. 414–419] for an alternative solution.

4.2.15 $\sum_{n=1}^{\infty}\frac{H_n^2H_n^{(2)}}{n^2}$

Show that

$$\sum_{n=1}^{\infty}\frac{H_n^2H_n^{(2)}}{n^2}=\frac{41}{12}\zeta(6)+2\zeta^2(3). \tag{4.61}$$

Solution.

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{H_n^4+3H_n^2H_n^{(2)}+2H_nH_n^{(3)}}{n^2}&=\sum_{n=1}^{\infty}\frac{H_n}{n}\left(\frac{H_n^3+3H_nH_n^{(2)}+2H_n^{(3)}}{n}\right)\\
&\qquad\{\text{recall the integral (2.62)}\}\\
&=\sum_{n=1}^{\infty}\frac{H_n}{n}\left(-\int_0^1 x^{n-1}\ln^3(1-x)\mathrm{d}x\right)\\
&=-\int_0^1\frac{\ln^3(1-x)}{x}\left(\sum_{n=1}^{\infty}\frac{H_n}{n}x^n\right)\mathrm{d}x\\
&\qquad\{\text{recall the generating function (2.6)}\}\\
&=-\int_0^1\frac{\ln^3(1-x)}{x}\left(\frac{1}{2}\ln^2(1-x)+\operatorname{Li}_2(x)\right)\mathrm{d}x\\
&\qquad\{\text{let } 1-x=y \text{ in the first integral}\}\\
&=-\frac{1}{2}\int_0^1\frac{\ln^5(y)}{1-y}\mathrm{d}y-\int_0^1\frac{\ln^3(1-x)\operatorname{Li}_2(x)}{x}\mathrm{d}x. \qquad (4.62)
\end{aligned}$$

Next, write $\frac{H_n}{n^2}=\int_0^1\frac{\zeta(2)-\operatorname{Li}_2(x)}{x}x^n\mathrm{d}x$ given in (3.66) to have

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{H_n^4-3H_n^2H_n^{(2)}+2H_nH_n^{(3)}}{n^2}&=\sum_{n=1}^{\infty}\left(H_n^3-3H_nH_n^{(2)}+2H_n^{(3)}\right)\frac{H_n}{n^2}\\
&=\sum_{n=1}^{\infty}\left(H_n^3-3H_nH_n^{(2)}+2H_n^{(3)}\right)\left(\int_0^1\frac{\zeta(2)-\operatorname{Li}_2(x)}{x}x^n\mathrm{d}x\right)\\
&=\int_0^1\frac{\zeta(2)-\operatorname{Li}_2(x)}{x}\left(\sum_{n=1}^{\infty}\left(H_n^3-3H_nH_n^{(2)}+2H_n^{(3)}\right)x^n\right)\mathrm{d}x\\
&\qquad\{\text{recall the generating function (2.22)}\}\\
&=\int_0^1\frac{\zeta(2)-\operatorname{Li}_2(x)}{x}\left(-\frac{\ln^3(1-x)}{1-x}\right)\mathrm{d}x\\
&=\int_0^1\frac{\ln^3(1-x)}{x}\left(\operatorname{Li}_2(x)-\zeta(2)\right)\mathrm{d}x+\underbrace{\int_0^1\frac{\ln^3(1-x)}{1-x}\left(\operatorname{Li}_2(x)-\zeta(2)\right)\mathrm{d}x}_{\text{IBP}}\\
&=\int_0^1\frac{\ln^3(1-x)\operatorname{Li}_2(x)}{x}\mathrm{d}x-\zeta(2)\int_0^1\frac{\ln^3(1-x)}{x}\mathrm{d}x-\frac{1}{4}\int_0^1\frac{\ln^5(1-x)}{x}\mathrm{d}x\\
&=\int_0^1\frac{\ln^3(1-x)\operatorname{Li}_2(x)}{x}\mathrm{d}x-\zeta(2)\int_0^1\frac{\ln^3(y)}{1-y}\mathrm{d}y-\frac{1}{4}\int_0^1\frac{\ln^5(y)}{1-y}\mathrm{d}y. \qquad (4.63)
\end{aligned}$$

Take the difference of (4.62) and (4.63),

$$\sum_{n=1}^{\infty}\frac{H_n^2H_n^{(2)}}{n^2}=\frac{1}{6}\zeta(2)\int_0^1\frac{\ln^3(y)}{1-y}\mathrm{d}y-\frac{1}{24}\int_0^1\frac{\ln^5(y)}{1-y}\mathrm{d}y$$
$$-\frac{1}{3}\int_0^1\frac{\ln^3(1-x)\operatorname{Li}_2(x)}{x}\mathrm{d}x.$$

The first two integrals are found in (3.8) and the last is given in (3.131).

4.2.16 $\sum_{n=1}^{\infty}\frac{H_n^4}{n^2}$

Show that

$$\sum_{n=1}^{\infty}\frac{H_n^4}{n^2}=\frac{979}{24}\zeta(6)+3\zeta^2(3). \tag{4.64}$$

Solution. Combine (4.62) and (4.63) then divide by 2,

$$\sum_{n=1}^{\infty}\frac{H_n^4}{n^2}=-\frac{1}{2}\zeta(2)\int_0^1\frac{\ln^3(y)}{1-y}\mathrm{d}y-\frac{3}{8}\int_0^1\frac{\ln^5(y)}{1-y}\mathrm{d}y-2\sum_{n=1}^{\infty}\frac{H_nH_n^{(3)}}{n^2}.$$

The sum on the RHS is given in (4.60) and the solution is finalized.
For different methods of computing (4.64) and (4.61), see [32, pp. 421–427].

4.2.17 $\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^3}$

Show that

$$\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^3}=-\frac{101}{48}\zeta(6)+\frac{5}{2}\zeta^2(3). \tag{4.65}$$

Solution.

$$\sum_{n=1}^{\infty}\frac{H_n^3+3H_nH_n^{(2)}+2H_n^{(3)}}{n^3}=\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\frac{H_n^3+3H_nH_n^{(2)}+2H_n^{(3)}}{n}\right)$$
$$\{\text{recall the integral (2.62)}\}$$
$$=\sum_{n=1}^{\infty}\frac{1}{n^2}\left(-\int_0^1 x^{n-1}\ln^3(1-x)\mathrm{d}x\right)$$

$$
\begin{aligned}
&= -\int_0^1 \frac{\ln^3(1-x)}{x}\left(\sum_{n=1}^{\infty}\frac{x^n}{n^2}\right)\mathrm{d}x \\
&= -\int_0^1 \frac{\ln^3(1-x)\operatorname{Li}_2(x)}{x}\mathrm{d}x. \qquad (4.66)
\end{aligned}
$$

On the other hand, write $\frac{1}{n^3}=\frac{1}{2}\int_0^1 x^{n-1}\ln^2(x)\mathrm{d}x$ to have

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\frac{H_n^3-3H_nH_n^{(2)}+2H_n^{(3)}}{n^3} \\
&= \sum_{n=1}^{\infty}\left(H_n^3-3H_nH_n^{(2)}+2H_n^{(3)}\right)\left(\frac{1}{2}\int_0^1 x^{n-1}\ln^2(x)\mathrm{d}x\right) \\
&= \frac{1}{2}\int_0^1 \frac{\ln^2(x)}{x}\left(\sum_{n=1}^{\infty}\left(H_n^3-3H_nH_n^{(2)}+2H_n^{(3)}\right)x^n\right)\mathrm{d}x \\
&\qquad \{\text{recall the generating function } (2.22)\} \\
&= \frac{1}{2}\int_0^1 \frac{\ln^2(x)}{x}\left(-\frac{\ln^3(1-x)}{1-x}\right)\mathrm{d}x \\
&\overset{1-x=y}{=} -\frac{1}{2}\int_0^1 \frac{\ln^3(y)\ln^2(1-y)}{y(1-y)}\mathrm{d}y \\
&\qquad \left\{\text{expand } \frac{\ln^2(1-y)}{1-y} \text{ in series given in } (2.11)\right\} \\
&= -\frac{1}{2}\sum_{n=1}^{\infty}\left(H_n^2-H_n^{(2)}\right)\int_0^1 y^{n-1}\ln^3(y)\mathrm{d}y \\
&= -\frac{1}{2}\sum_{n=1}^{\infty}\left(H_n^2-H_n^{(2)}\right)\left(-\frac{3!}{n^4}\right) \\
&= 3\sum_{n=1}^{\infty}\frac{H_n^2}{n^4}-3\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4}. \qquad (4.67)
\end{aligned}
$$

Take the difference of (4.66) and (4.67) then divide by 6,

$$
\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^3}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n^2}{n^4}-\frac{1}{6}\int_0^1\frac{\ln^3(1-x)\operatorname{Li}_2(x)}{x}\mathrm{d}x.
$$

The solution completes on gathering the results (4.56), (4.58), and (3.131).

For another approach, check [32, p. 411–414] .

4.2.18 $\sum_{n=1}^{\infty} \frac{H_n^3}{n^3}$

Show that

$$\sum_{n=1}^{\infty} \frac{H_n^3}{n^3} = \frac{93}{16}\zeta(6) - \frac{5}{2}\zeta^2(3). \tag{4.68}$$

Solution. Combine (4.63) and (4.67) then divide by 2,

$$\sum_{n=1}^{\infty} \frac{H_n^3}{n^3} = \frac{3}{2}\sum_{n=1}^{\infty} \frac{H_n^2}{n^4} - \frac{3}{2}\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^4} - 2\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^3}$$
$$-\frac{1}{2}\int_0^1 \frac{\ln^3(1-x)\operatorname{Li}_2(x)}{x}\mathrm{d}x.$$

These terms are calculated in (4.56), (4.58), (2.77), and (3.131).
A different solution may be found in [25].

4.2.19 $\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^5}$

Show that

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^5} = 5\zeta(2)\zeta(5) + 2\zeta(3)\zeta(4) - 10\zeta(7). \tag{4.69}$$

Solution. Set $a = 3$ in (1.131),

$$H_n^{(3)} = \zeta(3) - \sum_{k=1}^{\infty} \frac{1}{(n+k)^3}.$$

Divide both sides by n^4 then consider the summation,

$$\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^4} = \sum_{n=1}^{\infty} \frac{1}{n^4}\left(\zeta(3) - \sum_{k=1}^{\infty} \frac{1}{(n+k)^3}\right)$$
$$\{\text{distribute then change the order of summation}\}$$
$$= \zeta(3)\zeta(4) - \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty} \frac{1}{n^4(n+k)^3}\right)$$
$$\left\{\text{decompose } \frac{1}{n^4(n+k)^3} \text{ by partial fraction}\right\}$$

$$= \zeta(3)\zeta(4) - \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty} -\frac{10}{k^6}\left(\frac{1}{n} - \frac{1}{n+k}\right) + \frac{6}{k^5 n^2} + \frac{4}{k^5(n+k)^2} - \frac{3}{k^4 n^3}\right.$$
$$\left. + \frac{1}{k^4(n+k)^3} + \frac{1}{k^3 n^4}\right)$$

$$\left\{\text{use} \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+k}\right) = H_k \text{ given in (1.129) for the first term and use}\right\}$$
$$\left\{\sum_{n=1}^{\infty}\frac{1}{(k+n)^a} = \zeta(a) - H_k^{(a)} \text{ given in (1.127) for the third and fifth terms}\right\}$$

$$= \zeta(3)\zeta(4) - \sum_{k=1}^{\infty}\left(-\frac{10H_k}{k^6} + \frac{6\zeta(2)}{k^5} + 4\frac{\zeta(2) - H_k^{(2)}}{k^5} - \frac{3\zeta(3)}{k^4}\right.$$
$$\left. + \frac{\zeta(3) - H_k^{(3)}}{k^4} + \frac{\zeta(4)}{n^3}\right).$$

Upon rearranging the terms, the sum $\sum_{k=1}^{\infty}\frac{H_k^{(3)}}{k^4}$ cancels out from both sides,

$$\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^5} = \frac{5}{2}\zeta(2)\zeta(5) - \frac{1}{2}\zeta(3)\zeta(4) - \sum_{k=1}^{\infty}\frac{H_k}{k^6}.$$

The remaining sum is given in (4.4).
For an alternative solution, see (4.26).

4.2.20 $\sum_{n=1}^{\infty}\frac{H_n^2}{n^5}$

Show that

$$\sum_{n=1}^{\infty}\frac{H_n^2}{n^5} = 6\zeta(7) - \zeta(2)\zeta(5) - \frac{5}{2}\zeta(3)\zeta(4). \tag{4.70}$$

Solution.

$$\sum_{n=1}^{\infty}\frac{H_n^2 + H_n^{(2)}}{n^5} = \sum_{n=1}^{\infty}\frac{1}{n^4}\left(\frac{H_n^2 + H_n^{(2)}}{n}\right)$$
$$\{\text{recall the integral (2.62)}\}$$
$$= \sum_{n=1}^{\infty}\frac{1}{n^4}\left(\int_0^1 x^{n-1}\ln^2(1-x)\mathrm{d}x\right)$$

$$= \int_0^1 \frac{\ln^2(1-x)}{x}\left(\sum_{n=1}^{\infty}\frac{x^n}{n^4}\right)\mathrm{d}x$$

$$= \int_0^1 \frac{\ln^2(1-x)\,\mathrm{Li}_4(x)}{x}\mathrm{d}x$$

{expand $\ln^2(1-x)$ in series given in (2.7)}

$$= 2\sum_{n=1}^{\infty}\frac{H_{n-1}}{n}\int_0^1 x^{n-1}\,\mathrm{Li}_4(x)\mathrm{d}x$$

{recall the result (3.68) for the integral}

$$= 2\sum_{n=1}^{\infty}\left(\frac{H_n}{n}-\frac{1}{n^2}\right)\left(\frac{\zeta(4)}{n}-\frac{\zeta(3)}{n^2}+\frac{\zeta(2)}{n^3}-\frac{H_n}{n^4}\right).$$

Reorder the terms, we reach

$$3\sum_{n=1}^{\infty}\frac{H_n^2}{n^5} = -\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^5}+2\sum_{n=1}^{\infty}\frac{H_n}{n^6}+2\zeta(4)\sum_{n=1}^{\infty}\frac{H_n}{n^2}$$
$$-2\zeta(3)\sum_{n=1}^{\infty}\frac{H_n}{n^3}+2\zeta(2)\sum_{n=1}^{\infty}\frac{H_n}{n^4}-2\zeta(2)\zeta(5).$$

Gather the results (4.69) and (4.4) to end the solution.
Also check [32, p. 396–398] for a different method.

4.2.21 $\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^4}$

Show that

$$\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^4} = 18\zeta(7)-10\zeta(2)\zeta(5). \tag{4.71}$$

Solution. Divide both sides of (2.72):

$$\mathrm{Li}_3^2(x) = 12\sum_{n=1}^{\infty}\frac{H_n}{n^5}x^n+6\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4}x^n+2\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^3}x^n-20\,\mathrm{Li}_6(x)$$

by x then integrate using $\int_0^1 x^{n-1}\mathrm{d}x=\frac{1}{n}$,

$$12\sum_{n=1}^{\infty}\frac{H_n}{n^6}+6\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^5}+2\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^4}-20\zeta(7) = \int_0^1\frac{\mathrm{Li}_3^2(x)}{x}\mathrm{d}x$$

{expand $\mathrm{Li}_3(x)$ in series}

$$= \sum_{n=1}^{\infty} \frac{1}{n^3} \int_0^1 x^{n-1} \operatorname{Li}_3(x) \mathrm{d}x$$

$$\{\text{recall the result (3.67)}\}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^3} \left(\frac{\zeta(3)}{n} - \frac{\zeta(2)}{n^2} + \frac{H_n}{n^3} \right)$$

or

$$\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^4} = \frac{1}{2}\zeta(3)\zeta(4) - \frac{1}{2}\zeta(2)\zeta(5) + 10\zeta(7) - \frac{11}{2} \sum_{n=1}^{\infty} \frac{H_n}{n^6} - 3 \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^5}.$$

These two series are given in (4.4) and (4.69) respectively .

4.2.22 $\sum_{n=1}^{\infty} \frac{H_n^{(4)}}{n^3}$

Show that

$$\sum_{n=1}^{\infty} \frac{H_n^{(4)}}{n^3} = \zeta(3)\zeta(4) + 10\zeta(2)\zeta(5) - 17\zeta(7). \tag{4.72}$$

Solution. Setting $p = 3$ and $q = 4$ in (2.76),

$$\sum_{n=1}^{\infty} \frac{H_n^{(4)}}{n^3} = \zeta(4)\zeta(3) + \zeta(7) \sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^4}.$$

This sum is computed in (4.71).

4.2.23 $\sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(2)}}{n^3}$

Show that

$$\sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(2)}}{n^3} = \frac{19}{2}\zeta(3)\zeta(4) - 2\zeta(2)\zeta(5) - 7\zeta(7). \tag{4.73}$$

Solution.

$$\sum_{n=1}^{\infty} \frac{H_n^4 + 6H_n^2 H_n^{(2)} + 8H_n H_n^{(3)} + 2(H_n^{(2)})^2 + 6H_n^{(4)}}{n^3}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\frac{H_n^4 + 6H_n^2 H_n^{(2)} + 8H_n H_n^{(3)} + 2(H_n^{(2)})^2 + 6H_n^{(4)}}{n} \right)$$

$$\{\text{recall the integral (2.62)}\}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\int_0^1 x^{n-1} \ln^4(1-x) \mathrm{d}x \right)$$

$$= \int_0^1 \frac{\ln^4(1-x)}{x} \left(\sum_{n=1}^{\infty} \frac{x^n}{n^2} \right) \mathrm{d}x$$

$$= \int_0^1 \frac{\ln^4(1-x) \operatorname{Li}_2(x)}{x} \mathrm{d}x. \tag{4.74}$$

To establish another relation, multiply both sides of (2.27):

$$\sum_{n=1}^{\infty} \left(H_n^4 - 6H_n^2 H_n^{(2)} + 8H_n H_n^{(3)} + 3\left(H_n^{(2)}\right)^2 - 6H_n^{(4)} \right) x^n = \frac{\ln^4(1-x)}{1-x}$$

by $\frac{\ln^2(x)}{2x}$ then integrate using $\frac{1}{2}\int_0^1 x^{n-1} \ln^2(x) \mathrm{d}x = \frac{1}{n^3}$, we obtain

$$\sum_{n=1}^{\infty} \frac{H_n^4 - 6H_n^2 H_n^{(2)} + 8H_n H_n^{(3)} + 2(H_n^{(2)})^2 - 6H_n^{(4)}}{n^3}$$

$$= \frac{1}{2} \int_0^1 \frac{\ln^2(x) \ln^4(1-x)}{x(1-x)} \mathrm{d}x$$

$$\overset{1-x=y}{=} \frac{1}{2} \int_0^1 \frac{\ln^2(1-y) \ln^4(y)}{y(1-y)} \mathrm{d}y$$

$$\left\{ \text{expand } \frac{\ln^2(1-y)}{1-y} \text{ in series given in (2.11)} \right\}$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \left(H_n^2 - H_n^{(2)} \right) \int_0^1 y^{n-1} \ln^4(y) \mathrm{d}y$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \left(H_n^2 - H_n^{(2)} \right) \left(\frac{4!}{n^5} \right)$$

$$= 12 \sum_{n=1}^{\infty} \frac{H_n^2}{n^5} - 12 \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^5}. \tag{4.75}$$

Take the difference of (4.74) and (4.75) then divide by 12,

$$\sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(2)}}{n^3} = \frac{1}{12} \int_0^1 \frac{\ln^4(1-x) \operatorname{Li}_2(x)}{x} \mathrm{d}x$$

$$-\sum_{n=1}^{\infty}\frac{H_n^{(4)}}{n^3}-\sum_{n=1}^{\infty}\frac{H_n^2}{n^5}+\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^5}.$$

These terms are given in (3.69), (4.72), (4.70), and (4.69) respectively. Check [32, p. 456] for another solution.

4.2.24 $\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^7}$

Show that

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^7}=7\zeta(2)\zeta(7)+2\zeta(3)\zeta(6)+4\zeta(4)\zeta(5)-\frac{35}{2}\zeta(9). \tag{4.76}$$

Solution. Set $a=2$ in (1.131),

$$H_n^{(2)}=\zeta(2)-\sum_{k=1}^{\infty}\frac{1}{(n+k)^2}.$$

Divide both sides by n^7 then consider the summation,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^7}&=\sum_{n=1}^{\infty}\frac{1}{n^7}\left(\zeta(2)-\sum_{k=1}^{\infty}\frac{1}{(n+k)^2}\right)\\&=\zeta(2)\zeta(7)-\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{n^7(n+k)^2}\right)\\&=\zeta(2)\zeta(7)-\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{7}{k^8}\left(\frac{1}{n}-\frac{1}{n+k}\right)-\frac{6}{k^7n^2}-\frac{1}{k^7(n+k)^2}+\frac{5}{k^6n^3}\right.\\&\qquad\left.-\frac{4}{k^5n^4}+\frac{3}{k^4n^5}-\frac{2}{k^3n^6}+\frac{1}{k^2n^7}\right)\\&=\zeta(2)\zeta(7)-\sum_{k=1}^{\infty}\left(\frac{7H_k}{k^8}-\frac{6\zeta(2)}{k^7}-\frac{1}{k^7}\left(\zeta(2)-H_k^{(2)}\right)+\frac{5\zeta(3)}{k^6}\right.\\&\qquad\left.-\frac{4\zeta(4)}{k^5}+\frac{3\zeta(5)}{k^4}-\frac{2\zeta(6)}{k^3}+\frac{\zeta(7)}{k^2}\right)\\&=\zeta(2)\zeta(7)-7\sum_{k=1}^{\infty}\frac{H_k}{k^8}+6\zeta(2)\zeta(7)-\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^7}-3\zeta(3)\zeta(6)+\zeta(4)\zeta(5).\end{aligned}$$

Therefore,

$$2\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^7}=7\zeta(2)\zeta(7)-3\zeta(3)\zeta(6)+\zeta(4)\zeta(5)-7\sum_{k=1}^{\infty}\frac{H_k}{k^8}.$$

The latter sum is given in (4.4). Another approach may be found in (4.26).
Remark: For integers p and q, where $q>1$, $q\neq p$, and $p+q$ is even >6, there does not exist a closed form for the series $\sum_{n=1}^{\infty}\frac{H_n^{(p)}}{n^q}$.

4.3 Alternating Harmonic Series

4.3.1 $\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n}$

Show that

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n}=\frac{1}{2}\ln^2(2)-\frac{1}{2}\zeta(2). \tag{4.77}$$

Solution. Set $q=1$ in (4.11),

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n}&=-\int_0^1\frac{\ln(1+x)}{x(1+x)}\mathrm{d}x\\&\overset{x=\frac{y}{1-y}}{=}-\int_0^{\frac{1}{2}}\frac{\ln(1-y)}{y}\mathrm{d}y\\&=\operatorname{Li}_2(y)\Big|_0^{\frac{1}{2}}=-\operatorname{Li}_2\left(\frac{1}{2}\right),\end{aligned}$$

The value $\operatorname{Li}_2\left(\frac{1}{2}\right)=\frac{1}{2}\zeta(2)-\frac{1}{2}\ln^2(2)$ is given in (1.81).
For a different approach, set $x=-1$ in (2.6) then write $\operatorname{Li}_2(-1)=-\frac{1}{2}\zeta(2)$.

4.3.2 $\sum_{n=1}^{\infty}\frac{(-1)^n H_{2n}}{n}$

Show that

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_{2n}}{n}=-\frac{5}{8}\zeta(2)+\frac{1}{4}\ln^2(2). \tag{4.78}$$

Solution (i). Let $a_n = \frac{H_n}{n}$ in (1.10):

$$\sum_{n=1}^{\infty} (-1)^n a_{2n} = \Re \sum_{n=1}^{\infty} i^n a_n, \tag{4.79}$$

we get

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^n H_{2n}}{n} &= 2\sum_{n=1}^{\infty} \frac{(-1)^n H_{2n}}{2n} = 2\,\Re \sum_{n=1}^{\infty} \frac{i^n H_n}{n} \\ &\quad \{\text{set } x = i \text{ in the generating function (2.6)}\} \\ &= 2\,\Re \left\{ \operatorname{Li}_2(i) + \frac{1}{2}\ln^2(1-i) \right\}.\end{aligned}$$

The values of these two terms are given in (1.72) and (1.29).
Solution (ii). Set $x = 1$ in (2.41):

$$\sum_{n=1}^{\infty} (-1)^n \frac{2H_{2n} - H_n}{n} x^{2n} = -2\arctan^2(x),$$

we get

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^n H_{2n}}{n} &= \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n} - \arctan^2(1) \\ &\quad \{\text{recall the result (4.77)}\} \\ &= \frac{1}{2}\left(\frac{1}{2}\ln^2(2) - \frac{1}{2}\zeta(2) \right) - \frac{\pi^2}{16} \\ &= -\frac{5}{8}\zeta(2) + \frac{1}{4}\ln^2(2).\end{aligned}$$

4.3.3 $\sum_{n=1}^{\infty} \frac{(-1)^n H_{2n}}{n^2}$

Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_{2n}}{n^2} = \frac{23}{16}\zeta(3) - \pi G. \tag{4.80}$$

Solution. Applying the previous approach,

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_{2n}}{n^2} = 4\sum_{n=1}^{\infty} \frac{(-1)^n H_{2n}}{(2n)^2} = 4\,\Re \sum_{n=1}^{\infty} \frac{i^n H_n}{n^2}$$

$$\{\text{set } x = i \text{ in the generating function (2.8)}\}$$
$$= 4\Re\left\{\operatorname{Li}_3(i) - \operatorname{Li}_3(1-i) + \ln(1-i)\operatorname{Li}_2(1-i) + \frac{1}{2}\ln(i)\ln^2(1-i) + \zeta(3)\right\}.$$

By using the values (1.82), (1.29), and (1.30), we find:

$$\ln(i)\ln^2(1-i) = \frac{\pi^2}{8}\ln(2) - \left(\frac{\pi^3}{32} - \frac{\pi}{8}\ln^2(2)\right)i; \tag{4.81}$$

$$\ln(1-i)\operatorname{Li}_2(1-i) = -\frac{\pi}{4}G - \frac{\pi^2}{32}\ln(2) - \left(\frac{1}{2}\ln(2)G + \frac{\pi^3}{64} + \frac{\pi}{8}\ln^2(2)\right)i. \tag{4.82}$$

Collect these two values along with (1.73) to finish the solution.

4.3.4 $\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^2}$

Show that

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^2} = -\frac{5}{8}\zeta(3). \tag{4.83}$$

Solution (i). Put $q = 2$ in (4.11),

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^2} = \int_0^1 \frac{\ln(x)\ln(1+x)}{x(1+x)}\mathrm{d}x$$
$$= \int_0^1 \frac{\ln(x)\ln(1+x)}{x}\mathrm{d}x - \int_0^1 \frac{\ln(x)\ln(1+x)}{1+x}\mathrm{d}x.$$

These two integrals are calculated in (3.12) and (3.100).

Solution (ii). Setting $x = -1$ in (2.8),

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^2} = \operatorname{Li}_3(-1) - \operatorname{Li}_3(2) + \ln(2)\operatorname{Li}_2(2) + \frac{1}{2}\ln(-1)\ln^2(2) + \zeta(3).$$

The values of $\operatorname{Li}_3(2)$, and $\operatorname{Li}_2(2)$ are given in (1.113) and (1.112).

An alternative solution may be found in [32, pp. 508–509].

4.3.5 $\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(2)}}{n}$

Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(2)}}{n} = \frac{1}{2}\ln(2)\zeta(2) - \zeta(3). \tag{4.84}$$

Solution (i). Put $q = 2$ in (4.28),

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(2)}}{n} &= -\int_0^1 \frac{\ln(x)\ln\left(\frac{1+x}{2}\right)}{1-x}\mathrm{d}x \\ = -\int_0^1 \frac{\ln(x)\ln(1+x)}{1-x}\mathrm{d}x &+ \ln(2)\int_0^1 \frac{\ln(x)}{1-x}\mathrm{d}x.\end{aligned}$$

These two integrals are given in (3.103) and (3.8).

Solution (ii). Setting $x = -1$ in (2.68) produces

$$-\ln(2)\operatorname{Li}_2(-1) = 2\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n^2} + \sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(2)}}{n} - 3\operatorname{Li}_3(-1).$$

The first sum is found in (4.83).

Also check (4.27) for a different solution.

4.3.6 $\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(3)}}{n}$

Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(3)}}{n} = \frac{3}{4}\ln(2)\zeta(3) - \frac{19}{16}\zeta(4). \tag{4.85}$$

Solution. Setting $x = -1$ in (2.17),

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(3)}}{n} = \operatorname{Li}_4(-1) - \ln(2)\operatorname{Li}_3(-1) - \frac{1}{2}\operatorname{Li}_2^2(-1).$$

For another solution, see (4.27).

4.3.7 $\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n^3}$

Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n^3} = 2\operatorname{Li}_4\left(\frac{1}{2}\right) - \frac{11}{4}\zeta(4) + \frac{7}{4}\ln(2)\zeta(3) - \frac{1}{2}\ln^2(2)\zeta(2) + \frac{1}{12}\ln^4(2). \tag{4.86}$$

Solution (i). Put $q = 3$ in (4.11),

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n^3} = -\frac{1}{2}\int_0^1 \frac{\ln^2(x)\ln(1+x)}{x(1+x)}\mathrm{d}x$$

$$= -\frac{1}{2}\int_0^1 \frac{\ln^2(x)\ln(1+x)}{x}\mathrm{d}x + \frac{1}{2}\int_0^1 \frac{\ln^2(x)\ln(1+x)}{1+x}\mathrm{d}x.$$

These two integrals are given in (3.12) and (3.111).

Solution (ii). Expand $\ln(1-x)\ln(1+x)$ in series as given in (2.46),

$$\int_0^1 \frac{\ln(1-x)\ln(1+x)\ln(x)}{x}\mathrm{d}x$$

$$= \int_0^1 \frac{\ln(x)}{x}\left(-\sum_{n=1}^{\infty}\left(\frac{H_{2n}-H_n}{n} + \frac{1}{2n^2}\right)x^{2n}\right)\mathrm{d}x$$

$$= -\sum_{n=1}^{\infty}\left(\frac{H_{2n}-H_n}{n} + \frac{1}{2n^2}\right)\int_0^1 x^{2n-1}\ln(x)\mathrm{d}x$$

$$= -\sum_{n=1}^{\infty}\left(\frac{H_{2n}-H_n}{n} + \frac{1}{2n^2}\right)\left(-\frac{1}{(2n)^2}\right)$$

$$= 2\sum_{n=1}^{\infty}\frac{H_{2n}}{(2n)^3} - \frac{1}{4}\sum_{n=1}^{\infty}\frac{H_n}{n^3} + \frac{1}{8}\zeta(4)$$

{make use of (1.5) for the first sum}

$$= \sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^3} + \frac{3}{4}\sum_{n=1}^{\infty}\frac{H_n}{n^3} + \frac{1}{8}\zeta(4).$$

Collect the results (3.112) and (4.4) to finish the solution.

A different method is by putting $x = -1$ in (2.15). Also check [33].

4.3.8 $\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(2)}}{n^2}$

Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(2)}}{n^2} = -4\operatorname{Li}_4\left(\frac{1}{2}\right) + \frac{51}{16}\zeta(4) - \frac{7}{2}\ln(2)\zeta(3) + \ln^2(2)\zeta(2) - \frac{1}{6}\ln^4(2). \tag{4.87}$$

Solution. Put $x = -1$ in (2.69),

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(2)}}{n^2} = 3\operatorname{Li}_4(-1) + \frac{1}{2}\operatorname{Li}_2^2(-1) - 2\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n^3}.$$

Substituting the results (4.86) completes the solution.
Also see [32, pp. 505–506] for a different method.

4.3.9 $\sum_{n=1}^{\infty} \frac{(-1)^n H_n^2}{n^2}$

Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n^2}{n^2} = 2\operatorname{Li}_4\left(\frac{1}{2}\right) - \frac{41}{16}\zeta(4) + \frac{7}{4}\ln(2)\zeta(3) - \frac{1}{2}\ln^2(2)\zeta(2) + \frac{1}{12}\ln^4(2). \tag{4.88}$$

Solution.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^n H_n^2}{n^2} &= \sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n}\left(\frac{H_n}{n}\right) \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n}\left(-\int_0^1 x^{n-1}\ln(1-x)\mathrm{d}x\right) \\
&= -\int_0^1 \frac{\ln(1-x)}{x}\left(\sum_{n=1}^{\infty} \frac{H_n}{n}(-x)^n\right)\mathrm{d}x \\
&\qquad \{\text{recall the generating function (2.6)}\} \\
&= -\int_0^1 \frac{\ln(1-x)}{x}\left(\frac{1}{2}\ln^2(1+x) + \operatorname{Li}_2(-x)\right)\mathrm{d}x \\
&= -\frac{1}{2}\int_0^1 \frac{\ln(1-x)\ln^2(1+x)}{x}\mathrm{d}x - \int_0^1 \frac{\ln(1-x)\operatorname{Li}_2(-x)}{x}\mathrm{d}x.
\end{aligned}$$

The first integral is found in (3.115). For the second, expand $\operatorname{Li}_2(-x)$ in series:

$$\begin{aligned}
\int_0^1 \frac{\ln(1-x)\operatorname{Li}_2(-x)}{x}\mathrm{d}x &= \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left(\int_0^1 x^{n-1}\ln(1-x)\mathrm{d}x\right)\\
&= -\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left(-\frac{H_n}{n}\right)\\
&= \sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^3}.
\end{aligned}$$

This sum is calculated in (4.86) and the solution is finalized.

Also check [32, pp. 506–508] for a different solution.

4.3.10 $\sum_{n=1}^{\infty}\frac{(-1)^n H_n H_n^{(2)}}{n}$

Show that

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n H_n^{(2)}}{n} = -2\operatorname{Li}_4\left(\frac{1}{2}\right)+\zeta(4)-\frac{7}{8}\ln(2)\zeta(3)+\frac{1}{4}\ln^2(2)\zeta(2) -\frac{1}{12}\ln^4(2). \tag{4.89}$$

Solution. Using the identity (2.62), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}(-1)^n\frac{H_n^3+3H_nH_n^{(2)}+2H_n^{(3)}}{n}\\
&= \sum_{n=1}^{\infty}(-1)^n\left(-\int_0^1 x^{n-1}\ln^3(1-x)\mathrm{d}x\right)\\
&= \int_0^1 \ln^3(1-x)\left(\sum_{n=1}^{\infty}(-x)^{n-1}\right)\mathrm{d}x\\
&= \int_0^1 \frac{\ln^3(1-x)}{1+x}\mathrm{d}x\\
&\qquad \{\text{make use of (3.17)}\}\\
&= -6\operatorname{Li}_4\left(\frac{1}{2}\right).
\end{aligned} \tag{4.90}$$

On the other hand, write $\frac{1}{n}=\int_0^1 x^{n-1}\mathrm{d}x$,

$$\begin{aligned}&\sum_{n=1}^{\infty}(-1)^n\frac{H_n^3-3H_nH_n^{(2)}+2H_n^{(3)}}{n}\\&=\sum_{n=1}^{\infty}\left(H_n^3-3H_nH_n^{(2)}+2H_n^{(3)}\right)\left(\int_0^1 x^{n-1}\mathrm{d}x\right)\\&=\int_0^1\frac{1}{x}\left(\sum_{n=1}^{\infty}\left(H_n^3-3H_nH_n^{(2)}+2H_n^{(3)}\right)(-x)^n\right)\mathrm{d}x\\&\qquad\{\text{recall the generating function (2.22)}\}\\&=\int_0^1\frac{1}{x}\left(-\frac{\ln^3(1+x)}{1+x}\right)\mathrm{d}x\\&=\int_0^1\frac{\ln^3(1+x)}{1+x}\mathrm{d}x-\int_0^1\frac{\ln^3(1+x)}{x}\mathrm{d}x\\&\qquad\{\text{the first integral is }\ln^4(2)/4\text{ and the second one is given in (3.20)}\}\\&=6\operatorname{Li}_4\left(\frac{1}{2}\right)-6\zeta(4)+\frac{21}{4}\ln(2)\zeta(3)-\frac{3}{2}\ln^2(2)\zeta(2)+\frac{1}{2}\ln^4(2).\end{aligned}\tag{4.91}$$

Take the difference of (4.90) and (4.91) then divide by 6 to finalize the solution.

4.3.11 $\sum_{n=1}^{\infty}\frac{(-1)^nH_n^3}{n}$

Show that

$$\sum_{n=1}^{\infty}\frac{(-1)^nH_n^3}{n}=-\frac{5}{8}\zeta(4)+\frac{9}{8}\ln(2)\zeta(3)-\frac{3}{4}\ln^2(2)\zeta(2)+\frac{1}{4}\ln^4(2).\tag{4.92}$$

Solution. Adding (4.90) and (4.91),

$$\begin{aligned}&2\sum_{n=1}^{\infty}\frac{(-1)^nH_n^3}{n}+4\sum_{n=1}^{\infty}\frac{(-1)^nH_n^{(3)}}{n}\\&=-6\zeta(4)+\frac{21}{4}\ln(2)\zeta(3)-\frac{3}{2}\ln^2(2)\zeta(2)+\frac{1}{2}\ln^4(2).\end{aligned}$$

The second sum on the LHS is calculated in (4.85).

4.3.12 $\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n^4}$

Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n^4} = \frac{1}{2}\zeta(2)\zeta(3) - \frac{59}{32}\zeta(5). \tag{4.93}$$

Solution. Putting $q = 4$ in (4.11) gives

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n^4} &= \frac{1}{6}\int_0^1 \frac{\ln^3(x)\ln(1+x)}{x(1+x)}\mathrm{d}x \\ &\qquad \{\text{recall the result (3.122)}\} \\ &= \frac{1}{16}\int_0^1 \frac{\ln^4(t)}{1-t}\mathrm{d}t - \frac{1}{12}\int_0^1 \frac{\ln^3(t)\ln(1-t)}{1-t}\mathrm{d}t - \frac{5}{48}\int_0^1 \frac{\ln^4(x)}{1+x}\mathrm{d}x.\end{aligned}$$

These integrals are calculated in (3.8), (3.41), and (3.11) respectively.
For another method, see [33].

4.3.13 $\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(2)}}{n^3}$

Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(2)}}{n^3} = \frac{11}{32}\zeta(5) - \frac{5}{8}\zeta(2)\zeta(3). \tag{4.94}$$

Solution. Replace x with $-x$ in (2.69),

$$\operatorname{Li}_2^2(-x) = 4\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n^3}x^n + 2\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(2)}}{n^2}x^n - 6\operatorname{Li}_4(-x).$$

Divide both sides by x then integrate using $\int_0^1 x^{n-1}\mathrm{d}x = \frac{1}{n}$,

$$\int_0^1 \frac{\operatorname{Li}_2^2(-x)}{x}\mathrm{d}x = 4\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n^4} + 2\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(2)}}{n^3} + \frac{45}{8}\zeta(5).$$

Substitute the relation involving this integral from (3.129),

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(2)}}{n^3} = \frac{45}{16}\zeta(5) + \frac{5}{16}\zeta(2)\zeta(3) + \frac{7}{16}\sum_{n=1}^{\infty} \frac{H_n}{n^4} - \sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n^4}. \tag{4.95}$$

These two series are computed in (4.4) and (4.93).

4.3.14 $\sum_{n=1}^{\infty} \frac{(-1)^n H_n^2}{n^3}$

Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n^2}{n^3} = -4\operatorname{Li}_5\left(\frac{1}{2}\right) - 4\ln(2)\operatorname{Li}_4\left(\frac{1}{2}\right) + \frac{19}{32}\zeta(5) + \frac{11}{8}\zeta(2)\zeta(3)$$
$$-\frac{7}{4}\ln^2(2)\zeta(3) + \frac{2}{3}\ln^3(2)\zeta(2) - \frac{2}{15}\ln^5(2). \tag{4.96}$$

Solution. Substitute $\frac{1}{n^3} = \frac{1}{2}\int_0^1 x^{n-1}\ln^2(x)\mathrm{d}x$,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}\left(H_n^2 - H_n^{(2)} - \frac{2H_n}{n} + \frac{2}{n^2}\right)$$
$$= \sum_{n=1}^{\infty} (-1)^{n-1}\left(H_n^2 - H_n^{(2)} - \frac{2H_n}{n} + \frac{2}{n^2}\right)\left(\frac{1}{2}\int_0^1 x^{n-1}\ln^2(x)\mathrm{d}x\right)$$
$$= \frac{1}{2}\int_0^1 \frac{\ln^2(x)}{x}\left(\sum_{n=1}^{\infty}\left(H_n^2 - H_n^{(2)} - \frac{2H_n}{n} + \frac{2}{n^2}\right)(-x)^{n-1}\right)\mathrm{d}x$$
$$\{\text{recall the generating function (2.12)}\}$$
$$= \frac{1}{2}\int_0^1 \frac{\ln^2(x)}{x}\left(\frac{\ln^2(1+x)}{1+x}\right)$$
$$\overset{\text{IBP}}{=} \frac{1}{3}\int_0^1 \frac{\ln(x)\ln^3(1+x)}{x}\mathrm{d}x.$$

Distribute then rearrange the terms,

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n^2}{n^3} = -2\operatorname{Li}_5(-1) + \sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(2)}}{n^3} + 2\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n^4}$$
$$+\frac{1}{3}\int_0^1 \frac{\ln(x)\ln^3(1+x)}{x}\mathrm{d}x$$
$$\{\text{recall the relation involving these two sums from (4.95)}\}$$
$$= -2\operatorname{Li}_5(-1) + \frac{45}{16}\zeta(5) + \frac{5}{16}\zeta(2)\zeta(3) + \frac{7}{16}\sum_{n=1}^{\infty} \frac{H_n}{n^4} + \sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n^4}$$
$$+\frac{1}{3}\int_0^1 \frac{\ln(x)\ln^3(1+x)}{x}\mathrm{d}x.$$

These terms are calculated in (4.4), (4.93), and (3.125) respectively.

Check [32, pp. 517–519] for an alternative approach.

4.3.15 $\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(4)}}{n}$

Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(4)}}{n} = -2\zeta(5) + \frac{7}{8}\ln(2)\zeta(4) + \frac{3}{8}\zeta(2)\zeta(3). \tag{4.97}$$

Solution.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(4)}}{n} &= \sum_{n=1}^{\infty} (-1)^n H_n^{(4)} \left(\int_0^1 x^{n-1} \mathrm{d}x\right) \\
&= \int_0^1 \frac{1}{x}\left(\sum_{n=1}^{\infty} H_n^{(4)} (-x)^n\right) \mathrm{d}x \\
&\{\text{set } a = 4 \text{ in (2.2) to get this sum}\} \\
&= \int_0^1 \frac{1}{x}\left(\frac{\operatorname{Li}_4(-x)}{1+x}\right) \mathrm{d}x \\
&= \int_0^1 \frac{\operatorname{Li}_4(-x)}{x} \mathrm{d}x - \underbrace{\int_0^1 \frac{\operatorname{Li}_4(-x)}{1+x} \mathrm{d}x}_{\text{IBP}} \\
&= \operatorname{Li}_5(-1) - \ln(2)\operatorname{Li}_4(-1) + \int_0^1 \frac{\ln(1+x)\operatorname{Li}_3(-x)}{x} \mathrm{d}x \\
&\overset{\text{IBP}}{=} \operatorname{Li}_5(-1) - \ln(2)\operatorname{Li}_4(-1) - \operatorname{Li}_2(-1)\operatorname{Li}_3(-1) + \int_0^1 \frac{\operatorname{Li}_2^2(-x)}{x} \mathrm{d}x \\
&= -\frac{15}{16}\zeta(5) + \frac{7}{8}\ln(2)\zeta(4) - \frac{3}{8}\zeta(2)\zeta(3) + \int_0^1 \frac{\operatorname{Li}_2^2(-x)}{x} \mathrm{d}x \\
&\{\text{recall the relation involving this integral from (3.129)}\} \\
&= \frac{1}{4}\zeta(2)\zeta(3) + \frac{7}{8}\ln(2)\zeta(4) - \frac{15}{16}\zeta(5) + \frac{7}{8}\sum_{n=1}^{\infty} \frac{H_n}{n^4} + 2\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n^4}.
\end{aligned} \tag{4.98}$$

The solution finalizes on substituting the results (4.4) and (4.93).

For different approaches, see [32, p. 516] and (4.97).

4.3.16 $\sum_{n=1}^{\infty}\frac{(-1)^n H_n^{(3)}}{n^2}$

Show that

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n^{(3)}}{n^2}=\frac{21}{32}\zeta(5)-\frac{3}{4}\zeta(2)\zeta(3). \tag{4.99}$$

Solution.

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n H_n^{(3)}}{n^2}&=\sum_{n=1}^{\infty}\frac{(-1)^n H_n^{(3)}}{n}\left(\int_0^1 x^{n-1}\mathrm{d}x\right)\\&=\int_0^1\frac{1}{x}\left(\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n}(-x)^n\right)\mathrm{d}x\\&\qquad\{\text{recall the generating function}\}\\&=\int_0^1\frac{1}{x}\left(\operatorname{Li}_4(-x)-\ln(1+x)\operatorname{Li}_3(-x)-\frac{1}{2}\operatorname{Li}_2^2(-x)\right)\mathrm{d}x\\&=\int_0^1\frac{\operatorname{Li}_4(-x)}{x}\mathrm{d}x-\underbrace{\int_0^1\frac{\ln(1+x)\operatorname{Li}_3(-x)}{x}\mathrm{d}x}_{\text{IBP}}-\frac{1}{2}\int_0^1\frac{\operatorname{Li}_2^2(-x)}{x}\mathrm{d}x\\&=\operatorname{Li}_5(-1)+\operatorname{Li}_2(-1)\operatorname{Li}_3(-1)-\int_0^1\frac{\operatorname{Li}_2^2(-x)}{x}\mathrm{d}x-\frac{1}{2}\int_0^1\frac{\operatorname{Li}_2^2(-x)}{x}\mathrm{d}x\\&=\frac{3}{8}\zeta(2)\zeta(3)-\frac{15}{16}\zeta(5)-\frac{3}{2}\int_0^1\frac{\operatorname{Li}_2^2(-x)}{x}\mathrm{d}x.\end{aligned}$$

This integral is given in (3.128).

Check [32, pp. 513–515] for different methods for both series (4.99) and (4.94).

4.3.17 $\sum_{n=1}^{\infty}\frac{(-1)^n H_n H_n^{(2)}}{n^2}$

Show that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n H_n H_n^{(2)}}{n^2}&=4\operatorname{Li}_5\left(\frac{1}{2}\right)+4\ln(2)\operatorname{Li}_4\left(\frac{1}{2}\right)+\frac{7}{4}\ln^2(2)\zeta(3)\\&-\frac{23}{8}\zeta(5)-\frac{2}{3}\ln^3(2)\zeta(2)-\frac{15}{16}\zeta(2)\zeta(3)+\frac{2}{15}\ln^5(2).\end{aligned} \tag{4.100}$$

Solution.

$$\sum_{n=1}^{\infty}(-1)^n\frac{H_n^3+3H_nH_n^{(2)}+2H_n^{(3)}}{n^2}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\frac{H_n^3+3H_nH_n^{(2)}+2H_n^{(3)}}{n}\right)$$

{recall the integral (2.62)}

$$=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(-\int_0^1 x^{n-1}\ln^3(1-x)\mathrm{d}x\right)$$

$$=-\int_0^1\frac{\ln^3(1-x)}{x}\left(\sum_{n=1}^{\infty}\frac{(-x)^n}{n}\right)\mathrm{d}x$$

$$=\int_0^1\frac{\ln^3(1-x)\ln(1+x)}{x}\mathrm{d}x. \tag{4.101}$$

On the other hand,

$$\sum_{n=1}^{\infty}(-1)^n\frac{H_n^3-3H_nH_n^{(2)}+2H_n^{(3)}}{n^2}$$

$$=\sum_{n=1}^{\infty}(-1)^n\left(H_n^3-3H_nH_n^{(2)}+2H_n^{(3)}\right)\left(-\int_0^1 x^{n-1}\ln(x)\mathrm{d}x\right)$$

$$=-\int_0^1\frac{\ln(x)}{x}\left(\sum_{n=1}^{\infty}\left(H_n^3-3H_nH_n^{(2)}+2H_n^{(3)}\right)(-x)^n\right)\mathrm{d}x$$

{recall the generating function (2.22)}

$$=-\int_0^1\frac{\ln(x)}{x}\left(-\frac{\ln^3(1+x)}{1+x}\right)\mathrm{d}x$$

$$=\int_0^1\frac{\ln^3(1+x)\ln(x)}{x}\mathrm{d}x-\underbrace{\int_0^1\frac{\ln^3(1+x)\ln(x)}{1+x}\mathrm{d}x}_{\text{IBP}}$$

$$=\int_0^1\frac{\ln^3(1+x)\ln(x)}{x}\mathrm{d}x+\frac{1}{4}\int_0^1\frac{\ln^4(1+x)}{x}\mathrm{d}x. \tag{4.102}$$

Take the difference of the two relations in (4.101) and (4.102) then divide by 6,

$$\sum_{n=1}^{\infty}\frac{(-1)^nH_nH_n^{(2)}}{n^2}=\frac{1}{6}\int_0^1\frac{\ln^3(1-x)\ln(1+x)}{x}\mathrm{d}x$$

$$+\frac{1}{24}\int_0^1\frac{\ln^4(1+x)}{x}\mathrm{d}x+\frac{1}{6}\int_0^1\frac{\ln^3(1+x)\ln(x)}{x}\mathrm{d}x.$$

These three integrals are given in (3.123), (3.20), and (3.125) respectively.

4.3.18 $\sum_{n=1}^{\infty} \frac{(-1)^n H_n^3}{n^2}$

Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n^3}{n^2} = -6 \operatorname{Li}_5\left(\frac{1}{2}\right) - 6\ln(2)\operatorname{Li}_4\left(\frac{1}{2}\right) + \frac{9}{4}\zeta(5) + \frac{27}{16}\zeta(2)\zeta(3)$$
$$-\frac{21}{8}\ln^2(2)\zeta(3) + \ln^3(2)\zeta(2) - \frac{1}{5}\ln^5(2). \tag{4.103}$$

Solution. Combine (4.101) and (4.102) then divide by 2,

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n^3}{n^2} = \frac{1}{2}\int_0^1 \frac{\ln^3(1-x)\ln(1+x)}{x}\mathrm{d}x - \frac{1}{8}\int_0^1 \frac{\ln^4(1+x)}{x}\mathrm{d}x$$
$$-\frac{1}{2}\int_0^1 \frac{\ln^3(1+x)\ln(x)}{x}\mathrm{d}x - 2\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(3)}}{n^2}.$$

These terms are given in (3.123), (3.20), (3.125), and (4.99) respectively.

The two series (4.94) and (4.103) can be found computed differently in [32, pp. 520–523].

4.4 Harmonic Series with Powers of 2 in the Denominator

4.4.1 $\sum_{n=1}^{\infty} \frac{H_n}{n2^n}$

Show that

$$\sum_{n=1}^{\infty} \frac{H_n}{n2^n} = \frac{1}{2}\zeta(2). \tag{4.104}$$

Solution (i). Set $x = \frac{1}{2}$ in (2.6),

$$\sum_{n=1}^{\infty} \frac{H_n}{n2^n} = \operatorname{Li}_2\left(\frac{1}{2}\right) + \frac{1}{2}\ln^2(2),$$

where $\operatorname{Li}_2(\frac{1}{2}) = \frac{1}{2}\zeta(2) - \frac{1}{2}\ln^2(2)$ given in (1.81).

Solution (ii). Set $q = 0$ in (3.86),

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n}{n2^n} &= \operatorname{Li}_2\left(\frac{1}{2}\right) - \int_0^{\frac{1}{2}}\frac{\ln(1-x)}{1-x}\mathrm{d}x\\ &= \operatorname{Li}_2\left(\frac{1}{2}\right) + \frac{1}{2}\ln^2(2)\\ &= \frac{1}{2}\zeta(2).\end{aligned}$$

Solution (iii). Set $q = 0$ in (3.87),

$$\sum_{n=1}^{\infty}\frac{H_n}{n2^n} = \int_0^1 \frac{\ln(1+x)}{x}\mathrm{d}x$$

$$= -\operatorname{Li}_2(-x)\Big|_0^1 = -\operatorname{Li}_2(-1) = \frac{1}{2}\zeta(2).$$

4.4.2 $\sum_{n=1}^{\infty}\frac{H_n}{n^2 2^n}$

Show that

$$\sum_{n=1}^{\infty}\frac{H_n}{n^2 2^n} = \zeta(3) - \frac{1}{2}\ln(2)\zeta(2). \tag{4.105}$$

Solution (i). Set $x = \frac{1}{2}$ in (2.8),

$$\sum_{n=1}^{\infty}\frac{H_n}{n^2 2^n} = -\ln(2)\operatorname{Li}_2\left(\frac{1}{2}\right) - \frac{1}{2}\ln^3(2) + \zeta(3)\zeta(3) - \frac{1}{2}\ln(2)\zeta(2).$$

Solution (ii). Set $q = 1$ in (3.85),

$$\sum_{n=1}^{\infty}\frac{H_n}{n^2 2^n} = \int_0^{\frac{1}{2}}\frac{\ln(2x)\ln(1-x)}{x(1-x)}\mathrm{d}x$$

$$= \ln(2)\int_0^{\frac{1}{2}}\frac{\ln(1-x)}{1-x}\mathrm{d}x + \ln(2)\int_0^{\frac{1}{2}}\frac{\ln(1-x)}{x}\mathrm{d}x + \int_0^{\frac{1}{2}}\frac{\ln(x)\ln(1-x)}{x(1-x)}\mathrm{d}x$$

$\{$make use of (3.84) for the third integral$\}$

$$= -\frac{1}{2}\ln^3(2) - \ln(2)\operatorname{Li}_2\left(\frac{1}{2}\right) + \int_0^1\frac{\ln(x)\ln(1-x)}{x}\mathrm{d}x.$$

This integral is given in (3.9).

A different approach may be found in [32, p. 500].

4.4.3 $\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n2^n}$

Show that

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n2^n} = \frac{5}{8}\zeta(3). \tag{4.106}$$

Solution (i). Put $x = \frac{1}{2}$ in (2.68) then reorganize the terms,

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n2^n} = 3\operatorname{Li}_3\left(\frac{1}{2}\right) + \ln(2)\operatorname{Li}_2\left(\frac{1}{2}\right) - 2\sum_{n=1}^{\infty} \frac{H_n}{n^2 2^n}.$$

Collect the values (4.105), (1.81), and (1.92).
Solution (ii). Set $q = 1$ in (3.87),

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n2^n} = -\int_0^1 \frac{\ln(1-x)\ln(1+x)}{x}\mathrm{d}x.$$

This integral is given in (3.98).

4.4.4 $\sum_{n=1}^{\infty} \frac{H_n^2}{n2^n}$

Show that

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n2^n} = \frac{7}{8}\zeta(3). \tag{4.107}$$

Solution.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{H_n^2 + H_n^{(2)}}{n2^n} &= \sum_{n=1}^{\infty} \frac{1}{2^n}\left(\frac{H_n^2 + H_n^{(2)}}{n}\right) \\
&\quad \{\text{recall the integral (2.62)}\} \\
&= \sum_{n=1}^{\infty} \frac{1}{2^n}\left(\int_0^1 x^{n-1}\ln^2(1-x)\mathrm{d}x\right) \\
&= \int_0^1 \frac{\ln^2(1-x)}{x}\left(\sum_{n=1}^{\infty}\left(\frac{x}{2}\right)^n\right)\mathrm{d}x
\end{aligned}$$

$$\{\text{use the geometric series formula}\}$$
$$= \int_0^1 \frac{\ln^2(1-x)}{x}\left(\frac{\frac{x}{2}}{1-\frac{x}{2}}\right)\mathrm{d}x$$
$$\overset{1-x=y}{=} \int_0^1 \frac{\ln^2(y)}{1+y}\mathrm{d}y.$$

Substituting the results (3.11) and (4.106) completes the solution.
An alternative solution is by setting $x = \frac{1}{2}$ in the generating function (2.14).

4.4.5 $\sum_{n=1}^{\infty} \frac{H_n}{n^3 2^n}$

Show that

$$\sum_{n=1}^{\infty} \frac{H_n}{n^3 2^n} = \operatorname{Li}_4\left(\frac{1}{2}\right) + \frac{1}{8}\zeta(4) - \frac{1}{8}\ln(2)\zeta(3) + \frac{1}{24}\ln^4(2). \tag{4.108}$$

Solution. Set $a = 2$ in (3.86),

$$\sum_{n=1}^{\infty} \frac{H_n}{n^3 2^n} = \operatorname{Li}_4\left(\frac{1}{2}\right) - \frac{1}{2}\int_0^{\frac{1}{2}} \frac{\ln^2(2x)\ln(1-x)}{1-x}\mathrm{d}x$$
$$= \operatorname{Li}_4\left(\frac{1}{2}\right) - \frac{1}{2}\ln^2(2)\int_0^{\frac{1}{2}} \frac{\ln(1-x)}{1-x}\mathrm{d}x - \ln(2)\int_0^{\frac{1}{2}} \frac{\ln(x)\ln(1-x)}{1-x}\mathrm{d}x$$
$$-\frac{1}{2}\int_0^{\frac{1}{2}} \frac{\ln^2(x)\ln(1-x)}{1-x}\mathrm{d}x.$$

The last two integrals are computed in (3.99) and (3.110) respectively.
A different way is by setting $x = \frac{1}{2}$ in (2.15). Also check [32, pp. 500–501].

4.4.6 $\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2 2^n}$

Show that

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2 2^n} = \operatorname{Li}_4\left(\frac{1}{2}\right) + \frac{1}{16}\zeta(4) + \frac{1}{4}\ln(2)\zeta(3) - \frac{1}{4}\ln^2(2)\zeta(2) + \frac{1}{24}\ln^4(2). \tag{4.109}$$

Solution. let $x = \frac{1}{2}$ in (2.69),

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2 2^n} = 3\operatorname{Li}_4\left(\frac{1}{2}\right) - \frac{1}{2}\operatorname{Li}_2^2\left(\frac{1}{2}\right) - 2\sum_{n=1}^{\infty} \frac{H_n}{n^3 2^n}.$$

The sum on the RHS is computed in (4.108).

4.4.7 $\sum_{n=1}^{\infty} \frac{H_n^2}{n^2 2^n}$

Show that

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^2 2^n} = -\operatorname{Li}_4\left(\frac{1}{2}\right) + \frac{37}{16}\zeta(4) - \frac{7}{4}\ln(2)\zeta(3) + \frac{1}{4}\ln^2(2)\zeta(2) - \frac{1}{24}\ln^4(2). \tag{4.110}$$

Solution.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{H_n^2 + H_n^{(2)}}{n^2 2^n} &= \sum_{n=1}^{\infty} \frac{1}{n 2^n}\left(\frac{H_n^2 + H_n^{(2)}}{n}\right) \\
&\quad \{\text{recall the integral (2.62)}\} \\
&= \sum_{n=1}^{\infty} \frac{1}{n 2^n}\left(\int_0^1 x^{n-1}\ln^2(1-x)\mathrm{d}x\right) \\
&= \int_0^1 \frac{\ln^2(1-x)}{x}\left(\sum_{n=1}^{\infty} \frac{(\frac{x}{2})^n}{n}\right)\mathrm{d}x \\
&= -\int_0^1 \frac{\ln^2(1-x)\ln(1-\frac{x}{2})}{x}\mathrm{d}x \\
&\overset{1-x=y}{=} \int_0^1 \frac{\ln^2(y)\ln\left(\frac{2}{1+y}\right)}{1-y}\mathrm{d}y \\
&\quad \{\text{set } q = 3 \text{ in (3.47) to get this integral}\} \\
&= \frac{19}{8}\zeta(4) - \frac{3}{2}\ln(2)\zeta(3).
\end{aligned}$$

The second sum on the LHS is given in (4.109).

4.4.8 $\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n2^n}$

Show that

$$\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n2^n} = \operatorname{Li}_4\left(\frac{1}{2}\right) - \frac{5}{16}\zeta(4) + \frac{7}{8}\ln(2)\zeta(3) - \frac{1}{4}\ln^2(2)\zeta(2) + \frac{1}{24}\ln^4(2). \tag{4.111}$$

Solution (i). Take $x = \frac{1}{2}$ in the generating function (2.17),

$$\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n2^n} = \operatorname{Li}_4\left(\frac{1}{2}\right) + \ln(2)\operatorname{Li}_3\left(\frac{1}{2}\right) - \frac{1}{2}\operatorname{Li}_2^2\left(\frac{1}{2}\right),$$

where the values of $\operatorname{Li}_3(\frac{1}{2})$ and $\operatorname{Li}_2(\frac{1}{2})$ are given in (1.81) and (1.92).
Solution (ii). Put $q = 2$ in (3.87),

$$\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n2^n} = \frac{1}{2}\int_0^1 \frac{\ln^2(1-x)\ln(1+x)}{x}\mathrm{d}x.$$

This integral is given in (3.116).

4.4.9 $\sum_{n=1}^{\infty} \frac{H_n}{n^4 2^n}$

Show that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{H_n}{n^4 2^n} &= 2\operatorname{Li}_5\left(\frac{1}{2}\right) + \ln(2)\operatorname{Li}_4\left(\frac{1}{2}\right) - \frac{1}{6}\ln^3(2)\zeta(2) + \frac{1}{2}\ln^2(2)\zeta(3)\\ &-\frac{1}{8}\ln(2)\zeta(4) - \frac{1}{2}\zeta(2)\zeta(3) + \frac{1}{32}\zeta(5) + \frac{1}{40}\ln^5(2).\end{aligned} \tag{4.112}$$

Solution. Set $a = 3$ in (3.86),

$$\sum_{n=1}^{\infty} \frac{H_n}{n^4 2^n} = \operatorname{Li}_5\left(\frac{1}{2}\right) + \frac{1}{6}\int_0^{\frac{1}{2}} \frac{\ln^3(2x)\ln(1-x)}{1-x}\mathrm{d}x$$

$$= \operatorname{Li}_5\left(\frac{1}{2}\right) + \frac{1}{6}\ln^3(2)\int_0^{\frac{1}{2}} \frac{\ln(1-x)}{1-x}\mathrm{d}x + \frac{1}{2}\ln^2(2)\int_0^{\frac{1}{2}} \frac{\ln(x)\ln(1-x)}{1-x}\mathrm{d}x$$

$$+\frac{1}{2}\ln(2)\int_0^{\frac{1}{2}} \frac{\ln^2(x)\ln(1-x)}{1-x}\mathrm{d}x + \frac{1}{6}\int_0^{\frac{1}{2}} \frac{\ln^3(x)\ln(1-x)}{1-x}\mathrm{d}x.$$

The last three integrals are given in (3.99), (3.110), and (3.119) respectively. For a different approach, see [32, pp. 501–502].

4.4.10 $\sum_{n=1}^{\infty}\frac{H_n^{(4)}}{n2^n}$

The following sum is proposed by Cornel Vălean (see[37]):

$$\sum_{n=1}^{\infty}\frac{H_n^{(4)}}{n2^n}=6\operatorname{Li}_5\left(\frac{1}{2}\right)+6\ln(2)\operatorname{Li}_4\left(\frac{1}{2}\right)-\frac{81}{16}\zeta(5)-\frac{21}{8}\zeta(2)\zeta(3)$$
$$+\frac{21}{8}\ln^2(2)\zeta(3)-\ln^3(2)\zeta(2)+\frac{1}{5}\ln^5(2). \tag{4.113}$$

Solution. Put $q=3$ in (3.87), we obtain

$$\sum_{n=1}^{\infty}\frac{H_n^{(4)}}{n2^n}=-\frac{1}{6}\int_0^1\frac{\ln^3(1-x)\ln(1+x)}{x}\mathrm{d}x.$$

This integral is calculated in (3.123).

4.4.11 $\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^32^n}$

The following sum is proposed by Cornel Vălean (see [38]):

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^32^n}=-2\operatorname{Li}_5\left(\frac{1}{2}\right)-3\ln(2)\operatorname{Li}_4\left(\frac{1}{2}\right)+\frac{23}{64}\zeta(5)-\frac{1}{16}\ln(2)\zeta(4)$$
$$+\frac{23}{16}\zeta(2)\zeta(3)-\frac{23}{16}\ln^2(2)\zeta(3)+\frac{7}{12}\ln^3(2)\zeta(2)-\frac{13}{120}\ln^5(2). \tag{4.114}$$

Solution (i). Let $x=\frac{1}{2}$ in (2.73),

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^32^n}+\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^22^n}$$
$$=5\operatorname{Li}_5\left(\frac{1}{2}\right)+\ln(2)\operatorname{Li}_4\left(\frac{1}{2}\right)-2\sum_{n=1}^{\infty}\frac{H_n}{n^42^n}-\sum_{n=1}^{\infty}\frac{H_n^{(4)}}{n2^n}. \tag{4.115}$$

Now put $x=\frac{1}{2}$ in (2.71),

$$3\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3 2^n}+\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^2 2^n}$$
$$=\operatorname{Li}_2\left(\frac{1}{2}\right)\operatorname{Li}_3\left(\frac{1}{2}\right)+10\operatorname{Li}_5\left(\frac{1}{2}\right)-6\sum_{n=1}^{\infty}\frac{H_n}{n^4 2^n}. \tag{4.116}$$

Take the difference of (4.115) and (4.116),

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3 2^n}=\frac{5}{2}\operatorname{Li}_5\left(\frac{1}{2}\right)-\frac{1}{2}\ln(2)\operatorname{Li}_4\left(\frac{1}{2}\right)+\frac{1}{2}\operatorname{Li}_2\left(\frac{1}{2}\right)\operatorname{Li}_3\left(\frac{1}{2}\right)$$
$$-2\sum_{n=1}^{\infty}\frac{H_n}{n^4 2^n}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n^{(4)}}{n 2^n}.$$

The two sums on the RHS are given in (4.112) and (4.113) respectively.

Solution (ii). Divide both sides of (2.69) by x then integrate from $x=0$ to $\frac{1}{2}$, using $\int_0^{\frac{1}{2}} x^{n-1}\mathrm{d}x=\frac{1}{n2^n}$,

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3 2^n}=\frac{1}{2}\int_0^{\frac{1}{2}}\frac{\operatorname{Li}_2^2(x)}{x}\mathrm{d}x-2\sum_{n=1}^{\infty}\frac{H_n}{n^4 2^n}+3\operatorname{Li}_5\left(\frac{1}{2}\right).$$

These two terms are calculated in (3.130) and (4.112).

4.4.12 $\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^2 2^n}$

The following sum is also proposed by Cornel Vălean (see [38]):

$$\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^2 2^n}=4\operatorname{Li}_5\left(\frac{1}{2}\right)+3\ln(2)\operatorname{Li}_4\left(\frac{1}{2}\right)-\frac{81}{64}\zeta(5)+\frac{5}{16}\ln(2)\zeta(4)$$
$$-\frac{7}{8}\zeta(2)\zeta(3)+\frac{7}{8}\ln^2(2)\zeta(3)-\frac{5}{12}\ln^3(2)\zeta(2)+\frac{11}{120}\ln^5(2). \tag{4.117}$$

Solution. Combine (4.115) and (4.116),

$$\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^2 2^n}=-\frac{3}{2}\sum_{n=1}^{\infty}\frac{H_n^{(4)}}{n 2^n}+\frac{5}{2}\operatorname{Li}_5\left(\frac{1}{2}\right)+\frac{3}{2}\ln(2)\operatorname{Li}_4\left(\frac{1}{2}\right)$$
$$-\frac{1}{2}\operatorname{Li}_2\left(\frac{1}{2}\right)\operatorname{Li}_3\left(\frac{1}{2}\right).$$

Collect the value (4.113) to finalize the solution.

4.4.13 $\sum_{n=1}^{\infty} \frac{H_n^2}{n^3 2^n}$

Show that

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^3 2^n} = -2\operatorname{Li}_5\left(\frac{1}{2}\right) - \ln(2)\operatorname{Li}_4\left(\frac{1}{2}\right) + \frac{279}{64}\zeta(5) - \frac{37}{16}\ln(2)\zeta(4)$$
$$-\frac{9}{16}\zeta(2)\zeta(3) + \frac{7}{16}\ln^2(2)\zeta(3) + \frac{1}{12}\ln^3(2)\zeta(2) - \frac{1}{40}\ln^5(2). \qquad (4.118)$$

Solution. Multiply both sides of (2.11):

$$\sum_{n=1}^{\infty}(H_n^2 - H_n^{(2)})x^n = \frac{\ln^2(1-x)}{1-x}$$

by $\frac{\ln^2(x)}{x}$ then integrate from $x = 0$ to $\frac{1}{2}$,

$$\int_0^{\frac{1}{2}} \frac{\ln^2(1-x)\ln^2(x)}{x(1-x)}\mathrm{d}x = \sum_{n=1}^{\infty}\left(H_n^2 - H_n^{(2)}\right)\int_0^{\frac{1}{2}} x^{n-1}\ln^2(x)\mathrm{d}x$$

$$\overset{\text{IBP}}{=} \sum_{n=1}^{\infty}\left(H_n^2 - H_n^{(2)}\right)\left(\frac{\ln^2(2)}{n2^n} + \frac{2\ln(2)}{n^2 2^n} + \frac{2}{n^3 2^n}\right)$$

$$= \ln(2)\sum_{n=1}^{\infty}(H_n^2 - H_n^{(2)})\left(\frac{\ln(2)}{n2^n} + \frac{2}{n^2 2^n}\right) + 2\sum_{n=1}^{\infty}\frac{H_n^2 - H_n^{(2)}}{n^3 2^n}$$

$$\left\{\text{write } \frac{\ln(2)}{n2^n} + \frac{2}{n^2 2^n} = -\int_0^{\frac{1}{2}} x^{n-1}(\ln(2) + 2\ln(x))\mathrm{d}x\right\}$$

$$= \ln(2)\sum_{n=1}^{\infty}(H_n^{(2)} - H_n^2)\left(\int_0^{\frac{1}{2}} x^{n-1}(\ln(2) + 2\ln(x))\mathrm{d}x\right) + 2\sum_{n=1}^{\infty}\frac{H_n^2 - H_n^{(2)}}{n^3 2^n}$$

$$= \ln(2)\int_0^{\frac{1}{2}}\frac{\ln(2) + 2\ln(x)}{x}\left(\sum_{n=1}^{\infty}(H_n^{(2)} - H_n^2)x^n\right)\mathrm{d}x + 2\sum_{n=1}^{\infty}\frac{H_n^2 - H_n^{(2)}}{n^3 2^n}$$

{recall the generation function (2.11)}

$$= \ln(2)\int_0^{\frac{1}{2}}\frac{\ln(2) + 2\ln(x)}{x}\left(-\frac{\ln^2(1-x)}{1-x}\right)\mathrm{d}x + 2\sum_{n=1}^{\infty}\frac{H_n^2 - H_n^{(2)}}{n^3 2^n}.$$

Reorganizing the terms, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n^2}{n^3 2^n}&=\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3 2^n}+\frac{1}{2}\int_0^{\frac{1}{2}}\frac{\ln^2(1-x)\ln^2(x)}{x(1-x)}\mathrm{d}x\\&+\ln(2)\int_0^{\frac{1}{2}}\frac{\ln(x)\ln^2(1-x)}{x(1-x)}\mathrm{d}x+\frac{1}{2}\ln^2(2)\int_0^{\frac{1}{2}}\frac{\ln^2(1-x)}{x(1-x)}\mathrm{d}x.\end{aligned}\tag{4.119}$$

First integral: Set $q=2$ in (3.84),

$$\begin{aligned}\int_0^{\frac{1}{2}}\frac{\ln^2(1-x)\ln^2(x)}{x(1-x)}\mathrm{d}x&=\int_0^{1}\frac{\ln^2(1-x)\ln^2(x)}{x}\mathrm{d}x\\&\overset{\text{IBP}}{=}\frac{2}{3}\int_0^1\frac{\ln^3(x)\ln(1-x)}{1-x}\mathrm{d}x\\&\qquad\{\text{this integral is given in (3.41)}\}\\&=8\zeta(5)-4\zeta(2)\zeta(3).\end{aligned}$$

Second integral: Let $1-x=y$,

$$\begin{aligned}\int_0^{\frac{1}{2}}\frac{\ln(x)\ln^2(1-x)}{x(1-x)}\mathrm{d}x&=\underbrace{\int_{\frac{1}{2}}^1\frac{\ln(1-y)\ln^2(y)}{y}\mathrm{d}y}_{\text{IBP}}-\int_{\frac{1}{2}}^1\frac{\ln(1-y)\ln^2(y)}{1-y}\mathrm{d}y\\&=\frac{1}{3}\ln^4(2)+\frac{1}{3}\underbrace{\int_{\frac{1}{2}}^1\frac{\ln^3(y)}{1-y}\mathrm{d}y}_{\int_0^1-\int_0^{1/2}}-\underbrace{\int_{\frac{1}{2}}^1\frac{\ln(1-y)\ln^2(y)}{1-y}\mathrm{d}y}_{\int_0^1-\int_0^{1/2}}\\&=\frac{1}{3}\ln^4(2)+\frac{1}{3}\int_0^1\frac{\ln^3(y)}{1-y}\mathrm{d}y-\frac{1}{3}\int_0^{\frac{1}{2}}\frac{\ln^3(y)}{1-y}\mathrm{d}y-\int_0^1\frac{\ln(1-y)\ln^2(y)}{1-y}\mathrm{d}y\\&\qquad+\int_0^{\frac{1}{2}}\frac{\ln(1-y)\ln^2(y)}{1-y}\mathrm{d}y\\&\qquad\{\text{these integrals are given in (3.8), (3.19), (3.41), and (3.110)}\}\\&=2\operatorname{Li}_4\left(\frac{1}{2}\right)-\frac{9}{4}\zeta(4)+\frac{7}{4}\ln(2)\zeta(3)-\frac{1}{2}\ln^2(2)\zeta(2)+\frac{1}{12}\ln^4(2).\end{aligned}$$

Third integral: Let $1-x=y$,

$$\int_0^{\frac{1}{2}}\frac{\ln^2(1-x)}{x(1-x)}\mathrm{d}x=\int_{\frac{1}{2}}^1\frac{\ln^2(y)}{y}\mathrm{d}y+\underbrace{\int_{\frac{1}{2}}^1\frac{\ln^2(y)}{1-y}\mathrm{d}y}_{\int_0^1-\int_0^{1/2}}$$

$$= \frac{1}{3}\ln^3(2) + \int_0^1 \frac{\ln^2(y)}{1-y}\mathrm{d}y - \int_0^{\frac{1}{2}} \frac{\ln^2(y)}{1-y}\mathrm{d}y$$
$$\{\text{collect the results (3.8) and (3.19)}\}$$
$$= \frac{1}{4}\zeta(3).$$

Plug the three integrals along with (4.114) in (4.119) to complete the solution. For a different method, check [22].

4.4.14 $\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^2 2^n}$

The following sum is proposed by Cornel Vălean (see [39]):

$$\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^2 2^n} = 2\operatorname{Li}_5\left(\frac{1}{2}\right) + \ln(2)\operatorname{Li}_4\left(\frac{1}{2}\right) - \frac{31}{32}\zeta(5) + \frac{1}{8}\ln(2)\zeta(4)$$
$$+\frac{1}{8}\zeta(2)\zeta(3) - \frac{1}{12}\ln^3(2)\zeta(2) + \frac{1}{40}\ln^5(2). \tag{4.120}$$

Solution. Divide both sides of (2.62):

$$-\int_0^1 x^{n-1}\ln^3(1-x)\mathrm{d}x = \frac{H_n^3 + 3H_n H_n^{(2)} + 2H_n^{(3)}}{n}$$

by $n2^n$ then consider the summation over $n \geq 1$,

$$\sum_{n=1}^{\infty} \frac{H_n^3}{n^2 2^n} + 3\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^2 2^n} + 2\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^2 2^n}$$
$$= -\int_0^1 \frac{\ln^3(1-x)}{x}\left(\sum_{n=1}^{\infty} \frac{(x/2)^n}{n}\right)\mathrm{d}x$$
$$= -\int_0^1 \frac{\ln^3(1-x)\ln(1-x/2)}{x}\mathrm{d}x$$
$$\overset{1-x=y}{=} \int_0^1 \frac{\ln^3(y)\ln\left(\frac{2}{1+y}\right)}{1-y}\mathrm{d}y$$
$$\{\text{set } q = 4 \text{ in (3.47) to get this integral}\}$$
$$= 12\zeta(5) - \frac{21}{4}\ln(2)\zeta(4) - \frac{9}{4}\zeta(2)\zeta(3), \tag{4.121}$$

Next, divide both sides of (2.22):

$$-\frac{\ln^3(1-x)}{1-x}=\sum_{n=1}^{\infty}x^n\left(H_n^3-3H_nH_n^{(2)}+2H_n^{(3)}\right)$$

by x, then integrate using $\int_0^{\frac{1}{2}}x^{n-1}\mathrm{d}x=\frac{1}{n2^n}$,

$$\sum_{n=1}^{\infty}\frac{H_n^3-3H_nH_n^{(2)}+2H_n^{(3)}}{n2^n}=-\int_0^{\frac{1}{2}}\frac{\ln^3(1-x)}{x(1-x)}\mathrm{d}x. \tag{4.122}$$

On the other hand, multiply both sides of (2.22) by $-\frac{\ln(x)}{x}$ then integrate,

$$\begin{aligned}&\int_0^{\frac{1}{2}}\frac{\ln^3(1-x)\ln(x)}{x(1-x)}\mathrm{d}x\\ &=\sum_{n=1}^{\infty}\left(H_n^3-3H_nH_n^{(2)}+2H_n^{(3)}\right)\left(-\int_0^{\frac{1}{2}}x^{n-1}\ln(x)\mathrm{d}x\right)\\ &\overset{\text{IBP}}{=}\sum_{n=1}^{\infty}\left(H_n^3-3H_nH_n^{(2)}+2H_n^{(3)}\right)\left(\frac{\ln(2)}{n2^n}+\frac{1}{n^22^n}\right).\end{aligned}$$

Distribute then rearrange the terms,

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{H_n^3}{n^22^n}-3\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^22^n}+2\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^22^n}\\ &=\int_0^{\frac{1}{2}}\frac{\ln^3(1-x)\ln(x)}{x(1-x)}\mathrm{d}x-\ln(2)\sum_{n=1}^{\infty}\frac{H_n^3-3H_nH_n^{(2)}+2H_n^{(3)}}{n2^n}\\ &\left\{\text{the latter sum is equal to }-\int_0^{\frac{1}{2}}\frac{\ln^3(1-x)}{x(1-x)}\mathrm{d}x\text{ given in (4.122)}\right\}\\ &=\int_0^{\frac{1}{2}}\frac{\ln^3(1-x)\ln(x)}{x(1-x)}\mathrm{d}x+\ln(2)\int_0^{\frac{1}{2}}\frac{\ln^3(1-x)}{x(1-x)}\mathrm{d}x\\ &\overset{1-x\to x}{=}\int_{\frac{1}{2}}^{1}\frac{\ln^3(x)\ln(1-x)}{x(1-x)}\mathrm{d}x+\ln(2)\int_{\frac{1}{2}}^{1}\frac{\ln^3(x)}{x(1-x)}\mathrm{d}x.\end{aligned}$$

First integral:

$$\int_{\frac{1}{2}}^{1}\frac{\ln^3(x)\ln(1-x)}{x(1-x)}\mathrm{d}x$$

$$=\underbrace{\int_{\frac{1}{2}}^{1}\frac{\ln^3(x)\ln(1-x)}{x}\mathrm{d}x}_{\text{IBP}}-\int_{\frac{1}{2}}^{1}\frac{\ln^3(x)\ln(1-x)}{1-x}\mathrm{d}x$$

$$=\frac{1}{4}\ln^5(2)+\frac{1}{4}\underbrace{\int_{\frac{1}{2}}^{1}\frac{\ln^4(x)}{1-x}\mathrm{d}x}_{\int_0^1-\int_0^{1/2}}-\underbrace{\int_{\frac{1}{2}}^{1}\frac{\ln^3(x)\ln(1-x)}{1-x}\mathrm{d}x}_{\int_0^1-\int_0^{1/2}}$$

$$=\frac{1}{4}\ln^5(2)+\frac{1}{4}\int_0^1\frac{\ln^4(x)}{1-x}\mathrm{d}x-\frac{1}{4}\int_0^{\frac{1}{2}}\frac{\ln^4(x)}{1-x}\mathrm{d}x$$
$$-\int_0^1\frac{\ln^3(x)\ln(1-x)}{1-x}\mathrm{d}x+\int_0^{\frac{1}{2}}\frac{\ln^3(x)\ln(1-x)}{1-x}\mathrm{d}x$$

{these integrals are given in (3.8), (3.19), (3.41), and (3.119)}

$$=-12\operatorname{Li}_5\left(\frac{1}{2}\right)-12\ln(2)\operatorname{Li}_4\left(\frac{1}{2}\right)+\frac{285}{16}\zeta(5)-3\zeta(2)\zeta(3)$$
$$-\frac{21}{4}\ln^2(2)\zeta(3)+2\ln^3(2)\zeta(2)-\frac{2}{5}\ln^5(2).$$

Second integral:

$$\int_{\frac{1}{2}}^{1}\frac{\ln^3(x)}{x(1-x)}\mathrm{d}x=\int_{\frac{1}{2}}^{1}\frac{\ln^3(x)}{x}\mathrm{d}x+\underbrace{\int_{\frac{1}{2}}^{1}\frac{\ln^3(x)}{1-x}\mathrm{d}x}_{\int_0^1-\int_0^{1/2}}$$

$$=-\frac{1}{4}\ln^4(2)+\int_0^1\frac{\ln^3(x)}{1-x}\mathrm{d}x-\int_0^{\frac{1}{2}}\frac{\ln^3(x)}{1-x}\mathrm{d}x$$

{collect the results (3.8) and (3.19)}

$$=6\operatorname{Li}_4\left(\frac{1}{2}\right)-6\zeta(4)+\frac{21}{4}\ln(2)\zeta(3)-\frac{3}{2}\ln^2(2)\zeta(2)+\frac{1}{4}\ln^4(2).$$

Combine the results of the two integrals,

$$\sum_{n=1}^{\infty}\frac{H_n^3}{n^2 2^n}-3\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^2 2^n}+2\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^2 2^n}=-12\operatorname{Li}_5\left(\frac{1}{2}\right)-6\ln(2)\operatorname{Li}_4\left(\frac{1}{2}\right)$$
$$+\frac{285}{16}\zeta(5)-6\ln(2)\zeta(4)-3\zeta(2)\zeta(3)+\frac{1}{2}\ln^3(2)\zeta(2)-\frac{3}{20}\ln^5(2).\qquad(4.123)$$

Take the difference of (4.121) and (4.123) to finishes the solution.

4.4.15 $\sum_{n=1}^{\infty} \frac{H_n^3}{n^2 2^n}$

The following sum is proposed by Cornel Vălean (see [39]):

$$\sum_{n=1}^{\infty} \frac{H_n^3}{n^2 2^n} = -14\operatorname{Li}_5\left(\frac{1}{2}\right) - 9\ln(2)\operatorname{Li}_4\left(\frac{1}{2}\right) + \frac{279}{16}\zeta(5) - \frac{25}{4}\ln(2)\zeta(4)$$
$$-\frac{7}{8}\zeta(2)\zeta(3) - \frac{7}{4}\ln^2(2)\zeta(3) + \frac{13}{12}\ln^3(2)\zeta(2) - \frac{31}{120}\ln^5(2). \qquad (4.124)$$

Solution. Combining (4.121) and (4.123) yields

$$2\sum_{n=1}^{\infty} \frac{H_n^3}{n^2 2^n} + 4\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^2 2^n} = -12\operatorname{Li}_5\left(\frac{1}{2}\right) - 6\ln(2)\operatorname{Li}_4\left(\frac{1}{2}\right) + \frac{477}{16}\zeta(5)$$
$$-\frac{45}{4}\ln(2)\zeta(4) - \frac{21}{4}\zeta(2)\zeta(3) + \frac{1}{2}\ln^3(2)\zeta(2) - \frac{3}{20}\ln^5(2).$$

The sum on the LHS is calculated in (4.117).

4.5 Harmonic Series with Powers of $2n+1$ in the Denominator

4.5.1 $\sum_{n=0}^{\infty} \frac{(-1)^n H_{2n+1}}{2n+1}$

Show that

$$\sum_{n=0}^{\infty} \frac{(-1)^n H_{2n+1}}{2n+1} = G - \frac{\pi}{8}\ln(2). \qquad (4.125)$$

Solution (i). Set $a_n = \frac{H_n}{n}$ in (1.13), we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n H_{2n+1}}{2n+1} = \Im \sum_{n=1}^{\infty} \frac{i^n H_n}{n}$$
$$\{\text{let } x = i \text{ in the generating function (2.6)}\}$$
$$= \Im\left\{\operatorname{Li}_2(i) + \frac{1}{2}\ln^2(1-i)\right\}.$$

Gather the values (1.72) and (1.29) to finish the solution.

Solution (ii).

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-1)^n H_{2n+1}}{2n+1} &= \sum_{n=0}^{\infty}(-1)^n\left(-\int_0^1 x^{2n}\ln(1-x)\mathrm{d}x\right)\\ &= -\int_0^1 \ln(1-x)\left(\sum_{n=0}^{\infty}(-x^2)^n\right)\mathrm{d}x\\ &= -\int_0^1 \ln(1-x)\left(\frac{1}{1+x^2}\right)\mathrm{d}x.\end{aligned}$$

This integral is calculated in (3.97).

4.5.2 $\sum_{n=0}^{\infty}\frac{(-1)^n H_{2n+1}}{(2n+1)^2}$

Show that

$$\sum_{n=0}^{\infty}\frac{(-1)^n H_{2n+1}}{(2n+1)^2} = \Im\operatorname{Li}_3(1+i) - \frac{\pi}{16}\ln^2(2) - \frac{1}{2}\ln(2)G. \tag{4.126}$$

Solution. Let $a_n = \frac{H_n}{n^2}$ in (1.13),

$$\sum_{n=0}^{\infty}\frac{(-1)^n H_{2n+1}}{(2n+1)^2} = \Im\sum_{n=1}^{\infty}\frac{i^n H_n}{n^2}$$

$$\{\text{set } x = i \text{ in the generating function (2.8)}\}$$

$$= \Im\left\{\operatorname{Li}_3(i) - \operatorname{Li}_3(1-i) + \ln(1-i)\operatorname{Li}_2(1-i) + \frac{1}{2}\ln(i)\ln^2(1-i) + \zeta(3)\right\}.$$

Gather the values (1.73), (4.82), (1.30), and (1.29) to finish the solution.

4.5.3 $\sum_{n=0}^{\infty}\frac{(-1)^n H_n^{(2)}}{2n+1}$

Show that

$$\sum_{n=0}^{\infty}\frac{(-1)^n H_n^{(2)}}{2n+1} = 4\Im\operatorname{Li}_3(1-i) + \frac{5\pi^3}{48} + \frac{\pi}{4}\ln^2(2) + 2\ln(2)G. \tag{4.127}$$

Solution.

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{(-1)^n H_n^{(2)}}{2n+1} &= \sum_{n=0}^{\infty}(-1)^n H_n^{(2)}\int_0^1 x^{2n}\mathrm{d}x\\
&= \int_0^1\left(\sum_{n=0}^{\infty}H_n^{(2)}(-x^2)^n\right)\mathrm{d}x\\
&= \int_0^1 \frac{\operatorname{Li}_2(-x^2)}{1+x^2}\mathrm{d}x\\
&\overset{\text{IBP}}{=} -\frac{\pi^3}{48}+2\int_0^1\frac{\arctan x\ln(1+x^2)}{x}\mathrm{d}x\\
&\qquad\{\text{employ the generating function (2.45)}\}\\
&= -\frac{\pi^3}{48}-4\sum_{n=0}^{\infty}\frac{(-1)^n H_{2n}}{2n+1}\int_0^1 x^{2n}\mathrm{d}x\\
&= -\frac{\pi^3}{48}-4\sum_{n=0}^{\infty}\frac{(-1)^n H_{2n}}{(2n+1)^2}\\
&= -\frac{\pi^3}{48}-4\sum_{n=0}^{\infty}\frac{(-1)^n H_{2n+1}}{(2n+1)^2}+4\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}.
\end{aligned}$$

These two sums are given in (4.126) and (1.66) respectively.

4.5.4 $\sum_{n=0}^{\infty}\frac{(-1)^n H_{2n+1}^{(2)}}{2n+1}$

Show that

$$\sum_{n=0}^{\infty}\frac{(-1)^n H_{2n+1}^{(2)}}{2n+1} = 2\,\Im\operatorname{Li}_3(1-i)+\frac{17\pi^3}{192}+\frac{\pi}{8}\ln^2(2)+\frac{1}{2}\ln(2)G. \quad (4.128)$$

Solution. Set $a_n=\frac{H_n^{(2)}}{n}$ in (1.13),

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{(-1)^n H_{2n+1}^{(2)}}{2n+1} &= \Im\sum_{n=1}^{\infty}\frac{i^n H_n^{(2)}}{n}\\
&\qquad\{\text{employ the generating function (2.10)}\}\\
&= \Im\left\{\operatorname{Li}_3(i)+2\operatorname{Li}_3(1-i)-\ln(1-i)\operatorname{Li}_2(1-i)-\zeta(2)\ln(1-i)-2\zeta(3)\right\}.
\end{aligned}$$

These terms are given in (1.73), (4.82), and (1.29).

4.5.5 $\sum_{n=1}^{\infty} \frac{H_n}{(2n+1)^2}$

Show that

$$\sum_{n=1}^{\infty} \frac{H_n}{(2n+1)^2} = \frac{7}{4}\zeta(3) - \frac{3}{2}\ln(2)\zeta(2). \tag{4.129}$$

Solution. Set $q = 2$ in (4.31), we obtain

$$\sum_{n=1}^{\infty} \frac{H_n}{(2n+1)^2} = \int_0^1 \frac{\ln(x)\ln(1-x^2)}{1-x^2}\mathrm{d}x.$$

This integral is calculated in (3.102).
A different way to calculate this sum may be found in (3.45).

4.5.6 $\sum_{n=0}^{\infty} \frac{(-1)^n H_n}{(2n+1)^2}$

Show that

$$\sum_{n=0}^{\infty} \frac{(-1)^n H_n}{(2n+1)^2} = 2\,\Im\operatorname{Li}_3(1-i) + \frac{3\pi^3}{32} + \frac{\pi}{8}\ln^2(2) - \ln(2)G. \tag{4.130}$$

Solution. Write $H_{2n} = H_{2n+1} - \frac{1}{2n+1}$ in (3.1),

$$\ln(2) + H_n - H_{2n+1} + \frac{1}{2n+1} = \int_0^1 \frac{x^{2n}}{1+x}\mathrm{d}x.$$

Multiply both sides by $\frac{(-1)^n}{(2n+1)^2}$ then take the summation over $n \geq 0$,

$$\begin{aligned}
\ln(2)\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} + \sum_{n=0}^{\infty} \frac{(-1)^n H_n}{(2n+1)^2} - \sum_{n=0}^{\infty} \frac{(-1)^n H_{2n+1}}{(2n+1)^2} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3}& \\
= \int_0^1 \frac{1}{1+x}\left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)^2}\right)\mathrm{d}x& \\
= \int_0^1 \frac{1}{1+x}\left(\int_0^1 -\frac{\ln(y)}{1+x^2y^2}\mathrm{d}y\right)\mathrm{d}x& \\
\overset{xy=t}{=} \int_0^1 \int_0^x \frac{\ln(\frac{x}{t})}{x(1+x)(1+t^2)}\mathrm{d}t\,\mathrm{d}x& \\
\{\text{change the order of integration}\}&
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \frac{1}{1+t^2}\left(\int_t^1 \frac{\ln(\frac{x}{t})}{x(1+x)}\mathrm{d}x\right)\mathrm{d}t \\
&= \int_0^1 \frac{1}{1+t^2}\left(\operatorname{Li}_2(-t)+\frac{1}{2}\ln^2(t)+\ln(2)\ln(t)+\frac{\pi^2}{12}\right)\mathrm{d}t \\
&= \int_0^1 \frac{\operatorname{Li}_2(-t)}{1+t^2}\mathrm{d}t+\frac{1}{2}\int_0^1 \frac{\ln^2(t)}{1+t^2}\mathrm{d}t+\ln(2)\int_0^1 \frac{\ln(t)}{1+t^2}\mathrm{d}t+\frac{\pi^2}{12}\int_0^1 \frac{\mathrm{d}t}{1+t^2} \\
&\left\{\text{expand } \frac{1}{1+t^2} \text{ in series in the second and third integrals}\right\} \\
&= \int_0^1 \frac{\operatorname{Li}_2(-t)}{1+t^2}\mathrm{d}t+\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\int_0^1 t^{2n}\ln^2(t)\mathrm{d}t \\
&\quad +\ln(2)\sum_{n=0}^{\infty}(-1)^n\int_0^1 t^{2n}\ln(t)\mathrm{d}t+\frac{\pi^2}{12}\left(\frac{\pi}{4}\right) \\
&= \int_0^1 \frac{\operatorname{Li}_2(-t)}{1+t^2}\mathrm{d}t+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}-\ln(2)\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}+\frac{\pi^3}{48}.
\end{aligned}
$$

Reorganize the terms,

$$
\begin{aligned}
\sum_{n=0}^{\infty}\frac{(-1)^n H_n}{(2n+1)^2} &= \sum_{n=0}^{\infty}\frac{(-1)^n H_{2n+1}}{(2n+1)^2}-2\ln(2)\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} \\
&\quad +\int_0^1 \frac{\operatorname{Li}_2(-t)}{1+t^2}\mathrm{d}t+\frac{\pi^3}{48}. \qquad (4.131)
\end{aligned}
$$

Put together the results (4.126), (1.162), and (3.126) to end the solution.

4.5.7 $\sum_{n=0}^{\infty}\frac{(-1)^n H_{2n+1}}{(2n+1)^3}$

Show that

$$
\sum_{n=0}^{\infty}\frac{(-1)^n H_{2n+1}}{(2n+1)^3}=2\beta(4)-\frac{35\pi}{128}\zeta(3). \qquad (4.132)
$$

Solution. Using the series expansion of $\frac{\arctan x}{1+x^2}$ given in (2.43), we get

$$
\begin{aligned}
\int_0^1 \frac{x\ln^2(x)\arctan x}{1+x^2}\mathrm{d}x &= \int_0^1 x\ln^2(x)\left(\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(H_n-2H_{2n})x^{2n-1}\right)\mathrm{d}x \\
&= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(H_n-2H_{2n})\int_0^1 x^{2n}\ln^2(x)\mathrm{d}x
\end{aligned}
$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(H_n - 2H_{2n})\left(\frac{2}{(2n+1)^3}\right)$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n H_n}{(2n+1)^3} - 2\sum_{n=0}^{\infty}\frac{(-1)^n H_{2n}}{(2n+1)^3}$$

$$\left\{\text{use } H_{2n} = H_{2n+1} - \frac{1}{2n+1} \text{ in the last series}\right\}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n H_n}{(2n+1)^3} - 2\sum_{n=0}^{\infty}\frac{(-1)^n H_{2n+1}}{(2n+1)^3} + 2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^4}.$$

Rearrange the terms,

$$2\sum_{n=0}^{\infty}\frac{(-1)^n H_{2n+1}}{(2n+1)^3} = \sum_{n=0}^{\infty}\frac{(-1)^n H_n}{(2n+1)^3} + 2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^4} - \int_0^1 \frac{x\ln^2(x)\arctan x}{1+x^2}\mathrm{d}x$$

$$\left\{\text{write } 2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^4} = \int_0^1 \frac{\ln^2(x)\arctan x}{x}\mathrm{d}x,\right\}$$

$$\{\text{which follows from expanding } \arctan x \text{ in series then integrating}\}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n H_n}{(2n+1)^3} + \int_0^1 \frac{\ln^2(x)\arctan x}{x}\mathrm{d}x - \int_0^1 \frac{x\ln^2(x)\arctan x}{1+x^2}\mathrm{d}x$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n H_n}{(2n+1)^3} + \int_0^1 \frac{\ln^2(x)\arctan x}{x(1+x^2)}\mathrm{d}x.$$

Collect the results (4.37) and (3.118) to finish the solution.

For a different method, check the generalization (4.36).

4.5.8 $\sum_{n=0}^{\infty}\frac{(-1)^n H_{2n+1}^{(2)}}{(2n+1)^2}$

Show that

$$\sum_{n=0}^{\infty}\frac{(-1)^n H_{2n+1}^{(2)}}{(2n+1)^2} = -\beta(4) + \frac{35\pi}{64}\zeta(3) - \frac{\pi^2}{48}G. \tag{4.133}$$

Solution. Set $x = i$ in (2.69) then take the imaginary parts of both sides,

$$\Im \operatorname{Li}_2^2(i) = 4\Im\sum_{n=1}^{\infty}\frac{H_n}{n^3}x^i + 2\Im\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^2}x^i - 6\Im\operatorname{Li}_4(i),$$

use $\Im\sum_{n=1}^{\infty} i^n a_n = \sum_{n=0}^{\infty}(-1)^n a_{2n+1}$ given in (1.13),

$$\sum_{n=0}^{\infty}\frac{(-1)^n H_{2n+1}^{(2)}}{(2n+1)^2} = 3\Im\operatorname{Li}_4(i) + \frac{1}{2}\Im\operatorname{Li}_2^2(i) - 2\sum_{n=0}^{\infty}\frac{(-1)^n H_{2n+1}}{(2n+1)^3}.$$

Using the value (1.72), we have

$$\operatorname{Li}_2^2(i) = \frac{5}{128}\zeta(4) - G^2 - \left(\frac{\pi^2}{24}G\right)i.$$

Collect this result along with (1.74) and (4.132), the solution is finished.

4.5.9 $\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{(2n+1)^2}$

Show that

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{(2n+1)^2} = 8\operatorname{Li}_4\left(\frac{1}{2}\right) - \frac{121}{16}\zeta(4) + 7\ln(2)\zeta(3) - 2\ln^2(2)\zeta(2) + \frac{1}{3}\ln^4(2). \tag{4.134}$$

Solution (i).

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{(2n+1)^2} &= -\sum_{n=1}^{\infty}H_n^{(2)}\int_0^1 x^{2n}\ln(x)\mathrm{d}x\\
&= -\int_0^1 \ln(x)\left(\sum_{n=1}^{\infty}(x^2)^n H_n^{(2)}\right)\mathrm{d}x\\
&= -\int_0^1 \frac{\ln(x)\operatorname{Li}_2(x^2)}{1-x^2}\mathrm{d}x
\end{aligned}$$

$$\{\text{write } \operatorname{Li}_2(x^2) = 2\operatorname{Li}_2(x) + 2\operatorname{Li}_2(-x) \text{ given in (1.78)}\}$$

$$= -2\int_0^1\frac{\ln(x)\operatorname{Li}_2(x)}{1-x^2}\mathrm{d}x - 2\int_0^1\frac{\ln(x)\operatorname{Li}_2(-x)}{1-x^2}\mathrm{d}x$$

$$= -\int_0^1\frac{\ln(x)\operatorname{Li}_2(x)}{1-x}\mathrm{d}x - \int_0^1\frac{\ln(x)\operatorname{Li}_2(x)}{1+x}\mathrm{d}x - \int_0^1\frac{\ln(x)\operatorname{Li}_2(-x)}{1-x}\mathrm{d}x$$

$$-\int_0^1 \frac{\ln(x)\operatorname{Li}_2(-x)}{1+x}\mathrm{d}x.$$

First integral: Expand $\frac{\operatorname{Li}_2(x)}{1-x}$ using (2.3),

$$\begin{aligned}\int_0^1 \frac{\ln(x)\operatorname{Li}_2(x)}{1-x}\mathrm{d}x &= \sum_{n=1}^{\infty} H_{n-1}^{(2)} \int_0^1 x^{n-1}\ln(x)\mathrm{d}x\\ &= \sum_{n=1}^{\infty}\left(H_n^{(2)}-\frac{1}{n^2}\right)\left(-\frac{1}{n^2}\right)\\ &= -\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^2}+\zeta(4).\end{aligned}$$

Second integral: Expand $\frac{1}{1+x}$ in series,

$$\begin{aligned}\int_0^1 \frac{\ln(x)\operatorname{Li}_2(x)}{1+x}\mathrm{d}x &= -\sum_{n=1}^{\infty}(-1)^n\int_0^1 x^{n-1}\ln(x)\operatorname{Li}_2(x)\mathrm{d}x\\ &= -\sum_{n=1}^{\infty}(-1)^n\left(\frac{H_n^{(2)}}{n^2}+\frac{2H_n}{n^3}-\frac{2\zeta(2)}{n^2}\right)\\ &= -\sum_{n=1}^{\infty}\frac{(-1)^n H_n^{(2)}}{n^2}-2\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^3}-\frac{5}{2}\zeta(4).\end{aligned}$$

Third integral: Expand $\operatorname{Li}_2(-x)$ in series,

$$\begin{aligned}\int_0^1 \frac{\ln(x)\operatorname{Li}_2(-x)}{1-x}\mathrm{d}x &= \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\int_0^1 \frac{x^n\ln(x)}{1-x}\mathrm{d}x\\ &\quad \{\text{set } a=2 \text{ in (1.126) to get the integral}\}\\ &= \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left(H_n^{(2)}-\zeta(2)\right)\\ &= \sum_{n=1}^{\infty}\frac{(-1)^n H_n^{(2)}}{n^2}+\frac{5}{4}\zeta(4).\end{aligned}$$

Fourth integral: Expand $\frac{\operatorname{Li}_2(-x)}{1+x}$ using (2.3),

$$\begin{aligned}\int_0^1 \frac{\ln(x)\operatorname{Li}_2(-x)}{1+x}\mathrm{d}x &= -\sum_{k=1}^{\infty}(-1)^n H_{n-1}^{(2)}\int_0^1 x^{n-1}\ln(x)\mathrm{d}x\\ &= -\sum_{n=1}^{\infty}(-1)^n\left(H_n^{(2)}-\frac{1}{n^2}\right)\left(-\frac{1}{n^2}\right)\end{aligned}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(2)}}{n^2} + \frac{7}{8}\zeta(4).$$

Gathering the four integrals,

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{(2n+1)^2} = 2\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n^3} - \sum_{k=1}^{\infty} \frac{(-1)^n H_n^{(2)}}{n^2} + \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2} - \frac{5}{8}\zeta(4).$$

These sums are found in (4.86), (4.87), and (4.41) respectively.

Solution (ii). Set $p = q = 2$ in the third application of Abel's summation (2.79),

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{(2n+1)^2} = \frac{5}{8}\zeta(4) - \frac{1}{4}\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2} - \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(2)}}{n^2}$$

and these two sums are found in (4.87) and (4.41).

4.5.10 $\sum_{n=1}^{\infty} \frac{H_n^2}{(2n+1)^2}$

Show that

$$\sum_{n=1}^{\infty} \frac{H_n^2}{(2n+1)^2} = 8\,\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{61}{16}\zeta(4) + \ln^2(2)\zeta(2) + \frac{1}{3}\ln^4(2). \quad (4.135)$$

Solution. Write $\frac{1}{(2n+1)^2} = -\int_0^1 x^{2n}\ln(x)\mathrm{d}x$ to have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{H_n^2 - H_n^{(2)}}{(2n+1)^2} &= \sum_{n=1}^{\infty} (H_n^2 - H_n^{(2)}) \left(-\int_0^1 x^{2n}\ln(x)\mathrm{d}x\right) \\
&= -\int_0^1 \ln(x) \left(\sum_{n=1}^{\infty} (H_n^2 - H_n^{(2)}) x^{2n}\right) \mathrm{d}x \\
&\quad \{\text{replace } x \text{ with } x^2 \text{ in } (2.11)\} \\
&= -\int_0^1 \ln(x) \left(\frac{\ln^2(1-x^2)}{1-x^2}\right) \mathrm{d}x \\
&\overset{\sqrt{x}=y}{=} -\frac{1}{4}\int_0^1 \frac{\ln(y)\ln^2(1-y)}{\sqrt{y}(1-y)}\mathrm{d}y.
\end{aligned}$$

Collect the results (4.134) and (3.117) to complete the solution.

4.5.11 $\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{(2n+1)^3}$

Show that

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{(2n+1)^3} = \frac{49}{8}\zeta(2)\zeta(3) - \frac{93}{8}\zeta(5). \tag{4.136}$$

Solution. Set $q = 2$ and $p = 3$ in (2.79),

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{(2n+1)^3} = \frac{7}{8}\zeta(2)\zeta(3) - \frac{15}{8}\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^2} - 2\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(3)}}{n^2}.$$

These two sums are given in (4.47) and (4.99).

4.5.12 $\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{(2n+1)^2}$

Show that

$$\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{(2n+1)^2} = \frac{31}{2}\zeta(5) - 8\zeta(2)\zeta(3). \tag{4.137}$$

Solution. Set $q = 3$ and $p = 2$ in (2.79),

$$\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{(2n+1)^2} = \frac{3}{4}\zeta(2)\zeta(3) - \frac{15}{4}\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^3} - 4\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(2)}}{n^3}.$$

These two sums are given in (4.44) and (4.94).

4.5.13 $\sum_{n=0}^{\infty} \frac{H_n^{(3)}}{(2n+1)^3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(3)}}{n^3}$

Show that

$$\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{(2n+1)^3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(3)}}{n^3} = -\frac{17}{16}\zeta^2(3) - \frac{31}{16}\zeta(6). \tag{4.138}$$

Solution. Set $q = p = 3$ in (2.79),

$$\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{(2n+1)^3} = \frac{7}{8}\zeta^2(3) - \frac{31}{8}\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^3} - 4\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(3)}}{n^3}.$$

The first sum on the RHS is given in (2.77).

4.6 Skew Harmonic Series

4.6.1 $\sum_{n=1}^{\infty} \frac{(-1)^n \overline{H}_n}{n}$

Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n \overline{H}_n}{n} = -\frac{1}{2}\zeta(2) - \frac{1}{2}\ln^2(2). \tag{4.139}$$

Solution. Set $q = 1$ in (4.17),

$$\sum_{n=1}^{\infty} \frac{(-1)^n \overline{H}_n}{n} = \int_0^1 \frac{\ln(1-x)}{x(1+x)} \mathrm{d}x$$

$$= \underbrace{\int_0^1 \frac{\ln(1-x)}{x} \mathrm{d}x}_{1-x=y} - \int_0^1 \frac{\ln(1-x)}{1+x} \mathrm{d}x$$

$$= \int_0^1 \frac{\ln(y)}{1-y} \mathrm{d}y - \int_0^1 \frac{\ln(1-x)}{1+x} \mathrm{d}x.$$

Put together the results (3.8) and (3.17) to complete the solution.
For an alternative solution, set $x = -1$ in the generating function (2.30).

4.6.2 $\sum_{n=1}^{\infty} \frac{(-1)^n \overline{H}_n H_n}{n}$

Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n \overline{H}_n H_n}{n} = \frac{1}{3}\ln^3(2) - \ln(2)\zeta(2) - \frac{1}{4}\zeta(3). \tag{4.140}$$

Solution (i). Using the integral form of $\overline{H}_n$ given in (1.140), we have

$$\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_nH_n}{n}=\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n}\left(\ln(2)-\int_0^1\frac{(-x)^n}{1+x}\mathrm{d}x\right)$$

$$=\ln(2)\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n}-\int_0^1\frac{1}{1+x}\left(\sum_{n=1}^{\infty}\frac{H_n}{n}x^n\right)\mathrm{d}x$$

$$\{\text{recall the generating function (2.6)}\}$$

$$=\ln(2)\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n}-\int_0^1\frac{1}{1+x}\left(\frac{1}{2}\ln^2(1-x)+\operatorname{Li}_2(x)\right)\mathrm{d}x$$

$$=\ln(2)\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n}-\frac{1}{2}\int_0^1\frac{\ln^2(1-x)}{1+x}\mathrm{d}x-\underbrace{\int_0^1\frac{\operatorname{Li}_2(x)}{1+x}\mathrm{d}x}_{\text{IBP}}$$

$$=\ln(2)\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n}-\frac{1}{2}\int_0^1\frac{\ln^2(1-x)}{1+x}\mathrm{d}x$$

$$-\ln(2)\zeta(2)-\int_0^1\frac{\ln(1+x)\ln(1-x)}{x}\mathrm{d}x.$$

The solution finalizes on combining the results (4.77), (3.17), and (3.98).

Solution (ii).

$$\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_nH_n}{n}=\sum_{n=1}^{\infty}(-1)^n\overline{H}_n\left(\frac{H_n}{n}\right)$$

$$=\sum_{n=1}^{\infty}(-1)^n\overline{H}_n\left(-\int_0^1x^{n-1}\ln(1-x)\mathrm{d}x\right)$$

$$=-\int_0^1\frac{\ln(1-x)}{x}\left(\sum_{n=1}^{\infty}\overline{H}_n(-x)^n\right)\mathrm{d}x$$

$$\{\text{replace } x \text{ with } -x \text{ in (2.29)}\}$$

$$=-\int_0^1\frac{\ln(1-x)}{x}\left(\frac{\ln(1-x)}{1+x}\right)\mathrm{d}x$$

$$=\int_0^1\frac{\ln^2(1-x)}{1+x}\mathrm{d}x-\underbrace{\int_0^1\frac{\ln^2(1-x)}{x}\mathrm{d}x}_{1-x=y}$$

$$=\int_0^1\frac{\ln^2(1-x)}{1+x}\mathrm{d}x-\int_0^1\frac{\ln^2(y)}{1-y}\mathrm{d}y.$$

These two integrals are calculated in (3.17) and (3.8).

4.6.3 $\sum_{n=1}^{\infty} \frac{\overline{H}_n H_n}{n^2}$

Show that

$$\sum_{n=1}^{\infty} \frac{\overline{H}_n H_n}{n^2} = -3\operatorname{Li}_4\left(\frac{1}{2}\right) + \frac{43}{16}\zeta(4) + \frac{3}{4}\ln^2(2)\zeta(2) - \frac{1}{8}\ln^4(2). \quad (4.141)$$

The following solution is due to Sean Stewart and it may be found in [24]:

Solution.

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\overline{H}_n H_n}{n^2} &= \sum_{n=1}^{\infty} \frac{H_n}{n^2}\left(\ln(2) - \int_0^1 \frac{(-x)^n}{1+x}\mathrm{d}x\right)\\ &= \ln(2)\sum_{n=1}^{\infty}\frac{H_n}{n^2} - \int_0^1 \frac{1}{1+x}\left(\sum_{n=1}^{\infty}\frac{H_n}{n^2}(-x)^n\right)\mathrm{d}x\\ &\qquad \{\text{recall the generating function (2.8)}\}\\ &= \ln(2)\sum_{n=1}^{\infty}\frac{H_n}{n^2} - \int_0^1 \frac{\operatorname{Li}_3(-x)}{1+x}\mathrm{d}x + \int_0^1 \frac{\operatorname{Li}_3(1+x)}{1+x}\mathrm{d}x\\ &- \int_0^1 \frac{\ln(1+x)\operatorname{Li}_2(1+x)}{1+x}\mathrm{d}x - \frac{1}{2}\int_0^1 \frac{\ln(-x)\ln^2(1+x)}{1+x}\mathrm{d}x - \int_0^1 \frac{\zeta(3)}{1+x}\mathrm{d}x.\end{aligned}$$

First integral: Integrate by parts,

$$\begin{aligned}\int_0^1 \frac{\operatorname{Li}_3(-x)}{1+x}\mathrm{d}x &= \ln(1+x)\operatorname{Li}_3(-x)\Big|_0^1 - \int_0^1 \frac{\ln(1+x)\operatorname{Li}_2(-x)}{x}\mathrm{d}x\\ &= -\frac{3}{4}\ln(2)\zeta(3) + \frac{1}{2}\operatorname{Li}_2^2(-x)\Big|_0^1\\ &= -\frac{3}{4}\ln(2)\zeta(3) + \frac{5}{16}\zeta(4).\end{aligned}$$

Second integral:

$$\int_0^1 \frac{\operatorname{Li}_3(1+x)}{1+x}\mathrm{d}x = \operatorname{Li}_4(1+x)\Big|_0^1 = \operatorname{Li}_4(2) - \zeta(4).$$

Third integral: Integrate by parts,

$$\begin{aligned}&\int_0^1 \frac{\ln(1+x)\operatorname{Li}_2(1+x)}{1+x}\mathrm{d}x\\ &= \operatorname{Li}_3(1+x)\ln(1+x)\Big|_0^1 - \int \frac{\operatorname{Li}_3(1+x)}{1+x}\mathrm{d}x\end{aligned}$$

$$\begin{aligned}&= \ln(2)\operatorname{Li}_3(2) - \operatorname{Li}_4(1+x)\Big|_0^1\\ &= \ln(2)\operatorname{Li}_3(2) - \operatorname{Li}_4(2) + \zeta(4).\end{aligned}$$

Fourth integral: Integrate by parts twice,

$$\begin{aligned}&\int_0^1 \frac{\ln(-x)\ln^2(1+x)}{1+x}\mathrm{d}x\\ &= -\operatorname{Li}_2(1+x)\ln^2(1+x)\Big|_0^1 + 2\int_0^1 \frac{\operatorname{Li}_2(1+x)\ln(1+x)}{1+x}\mathrm{d}x\\ &= -\ln^2(2)\operatorname{Li}_2(2) + 2\ln(2)\operatorname{Li}_3(2) - 2\operatorname{Li}_4(2) + 2\zeta(4).\end{aligned}$$

Fifth integral:

$$\int_0^1 \frac{\zeta(3)}{1+x}\mathrm{d}x = \zeta(3)\ln(1+x)\Big|_0^1 = \ln(2)\zeta(3).$$

Group all five integrals along with the result (4.4), we have

$$\sum_{n=1}^{\infty}\frac{\overline{H}_n H_n}{n^2} = \frac{7}{4}\ln(2)\zeta(3) - \frac{53}{16}\zeta(4) + 3\operatorname{Li}_4(2) - 2\ln(2)\operatorname{Li}_3(2) + \frac{1}{2}\ln^2(2)\operatorname{Li}_2(2).$$

The values of $\operatorname{Li}_4(2)$, $\operatorname{Li}_3(2)$, and $\operatorname{Li}_2(2)$ are given in (1.114), (1.113), and (1.112).

4.6.4 $\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n H_n}{n^2}$

Show that

$$\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n H_n}{n^2} = 3\operatorname{Li}_4\left(\frac{1}{2}\right) - \frac{29}{16}\zeta(4) - \frac{3}{4}\ln^2(2)\zeta(2) + \frac{1}{8}\ln^4(2). \tag{4.142}$$

Solution.

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n H_n}{n^2} &= \sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^2}\left(\ln(2) - \int_0^1 \frac{(-x)^n}{1+x}\mathrm{d}x\right)\\ &= \ln(2)\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^2} - \int_0^1 \frac{1}{1+x}\left(\sum_{n=1}^{\infty}\frac{H_n}{n^2}x^n\right)\mathrm{d}x\\ &= \ln(2)\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^2} - \int_0^1 \frac{\operatorname{Li}_3(x)}{1+x}\mathrm{d}x + \int_0^1 \frac{\operatorname{Li}_3(1-x)}{1+x}\mathrm{d}x\end{aligned}$$

$$-\int_0^1 \frac{\ln(1-x)\operatorname{Li}_2(1-x)}{1+x}\mathrm{d}x - \frac{1}{2}\int_0^1 \frac{\ln(x)\ln^2(1-x)}{1+x}\mathrm{d}x - \int_0^1 \frac{\zeta(3)}{1+x}\mathrm{d}x.$$

First integral: Expand $\frac{1}{1+x}$ in series,

$$\int_0^1 \frac{\operatorname{Li}_3(x)}{1+x}\mathrm{d}x = -\sum_{n=1}^{\infty}(-1)^n \int_0^1 x^{n-1}\operatorname{Li}_3(x)\mathrm{d}x$$
$$\{\text{recall the result (3.67)}\}$$
$$= -\sum_{n=1}^{\infty}(-1)^n\left(\frac{\zeta(3)}{n} - \frac{\zeta(2)}{n^2} + \frac{H_n}{n^3}\right)$$
$$= \ln(2)\zeta(3) - \frac{5}{4}\zeta(4) - \sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^3}.$$

Second integral: Let $1-x=y$,

$$\int_0^1 \frac{\operatorname{Li}_3(1-x)}{1+x}\mathrm{d}x = \int_0^1 \frac{\operatorname{Li}_3(y)}{2-y}\mathrm{d}y$$
$$\left\{\text{expand } \frac{1}{2-y} \text{ in series as } \sum_{n=1}^{\infty}\frac{y^{n-1}}{2^n}\right\}$$
$$= \sum_{n=1}^{\infty}\frac{1}{2^n}\int_0^1 y^{n-1}\operatorname{Li}_3(y)\mathrm{d}y$$
$$= \sum_{n=1}^{\infty}\frac{1}{2^n}\left(\frac{\zeta(3)}{n} - \frac{\zeta(2)}{n^2} + \frac{H_n}{n^3}\right)$$
$$= \ln(2)\zeta(3) - \zeta(2)\operatorname{Li}_2\left(\frac{1}{2}\right) + \sum_{n=1}^{\infty}\frac{H_n}{2^n n^3}.$$

Third integral: Let $1-x=y$,

$$\int_0^1 \frac{\ln(1-x)\operatorname{Li}_2(1-x)}{1+x}\mathrm{d}x = \int_0^1 \frac{\ln(y)\operatorname{Li}_2(y)}{2-y}\mathrm{d}y$$
$$= \sum_{n=1}^{\infty}\frac{1}{2^n}\int_0^1 y^{n-1}\ln(y)\operatorname{Li}_2(y)\mathrm{d}y$$
$$= \sum_{n=1}^{\infty}\frac{1}{2^n}\int_0^1 \frac{\partial}{\partial n}y^{n-1}\operatorname{Li}_2(y)\mathrm{d}y$$
$$= \sum_{n=1}^{\infty}\frac{1}{2^n}\frac{d}{dn}\int_0^1 y^{n-1}\operatorname{Li}_2(y)\mathrm{d}y$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d}{dn} \left(\frac{\zeta(2)}{n} - \frac{H_n}{n^2} \right)$$

{use the derivative of the harmonic number given in (1.132)}

$$= \sum_{n=1}^{\infty} \frac{1}{2^n} \left(\frac{2H_n}{n^3} + \frac{H_n^{(2)}}{n^2} - \frac{2\zeta(2)}{n^2} \right)$$

$$= 2\sum_{n=1}^{\infty} \frac{H_n}{2^n n^3} + \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{2^n n^2} - 2\zeta(2) \operatorname{Li}_2\left(\frac{1}{2}\right). \tag{4.143}$$

Fourth integral: Let $1 - x = y$,

$$\int_0^1 \frac{\ln(x)\ln^2(1-x)}{1+x} \mathrm{d}x = \int_0^1 \frac{\ln(1-y)\ln^2(y)}{2-y} \mathrm{d}y$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^n} \int_0^1 y^{n-1} \ln(1-y)\ln^2(y) \mathrm{d}y$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d^2}{dn^2} \int_0^1 y^{n-1} \ln(1-y) \mathrm{d}y$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d^2}{dn^2} \left(-\frac{H_n}{n} \right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^n} \left(\frac{2\zeta(3)}{n} + \frac{2\zeta(2)}{n^2} - \frac{2H_n}{n^3} - \frac{2H_n^{(2)}}{n^2} - \frac{2H_n^{(3)}}{n} \right)$$

$$= 2\ln(2)\zeta(3) + 2\zeta(2) \operatorname{Li}_2\left(\frac{1}{2}\right) - 2\sum_{n=1}^{\infty} \frac{H_n}{2^n n^3} - 2\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{2^n n^2} - 2\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{2^n n}. \tag{4.144}$$

Combine all integrals then rearrange the terms,

$$\sum_{n=1}^{\infty} \frac{(-1)^n \overline{H}_n H_n}{n^2} = \frac{5}{4}\zeta(4) - 2\ln(2)\zeta(3)$$

$$+ \ln(2) \sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n^2} + \sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n^3} + \sum_{n=1}^{\infty} \frac{H_n^{(3)}}{2^n n}.$$

These three sums are calculated in (4.83), (4.86), and (4.111) respectively.

4.6.5 $\sum_{n=1}^{\infty} \frac{\overline{H}_{2n} H_{2n}}{n^2}$

Show that

$$\sum_{n=1}^{\infty} \frac{\overline{H}_{2n} H_{2n}}{n^2} = \frac{7}{4}\zeta(4). \tag{4.145}$$

Solution. Put $a_n = \frac{\overline{H}_n H_n}{n^2}$ in (1.5):

$$\sum_{n=1}^{\infty} a_{2n} = \frac{1}{2}\sum_{n=1}^{\infty} a_n + \frac{1}{2}\sum_{n=1}^{\infty} (-1)^n a_n,$$

we have

$$\sum_{n=1}^{\infty} \frac{\overline{H}_{2n} H_{2n}}{(2n)^2} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{\overline{H}_n H_n}{n^2} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^n \overline{H}_n H_n}{n^2}.$$

These two sums are given in (4.141) and (4.142).

4.7 Harmonic Series with Rational Argument

4.7.1 $\sum_{n=1}^{\infty} \frac{(-1)^n H_{\frac{n}{2}}}{n}$

Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_{\frac{n}{2}}}{n} = \ln^2(2) - \frac{1}{2}\zeta(2). \tag{4.146}$$

Solution (i). Replace n with $\frac{n}{2}$ in (3.2),

$$\int_0^1 x^{n-1} \ln(1+x) \mathrm{d}x = \frac{H_n - H_{\frac{n}{2}}}{n}. \tag{4.147}$$

Making use of this integral, we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} (-1)^n \frac{H_n - H_{\frac{n}{2}}}{n} &= \sum_{n=1}^{\infty} (-1)^n \left(\int_0^1 x^{n-1} \ln(1+x) \mathrm{d}x\right) \\
&= \int_0^1 \frac{\ln(1+x)}{x} \left(\sum_{n=1}^{\infty} (-x)^n\right) \mathrm{d}x \\
&= \int_0^1 \frac{\ln(1+x)}{x} \left(\frac{-x}{1+x}\right) \mathrm{d}x
\end{aligned}$$

$$= -\frac{1}{2}\ln^2(2).$$

The first sum of the LHS is given in (4.77).

Solution (ii). Let $y = x^2$ in the integral (1.124), we get

$$\frac{H_n}{n} = -\int_0^1 y^{n-1}\ln(1-y)\mathrm{d}y = -2\int_0^1 x^{2n-1}\ln(1-x^2)\mathrm{d}x.$$

Replace n with $\frac{n}{2}$,

$$\frac{H_{\frac{n}{2}}}{n} = -\int_0^1 x^{n-1}\ln(1-x^2)\mathrm{d}x. \tag{4.148}$$

Utilizing this integral, we get

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n H_{\frac{n}{2}}}{n} &= \sum_{n=1}^{\infty}(-1)^n\left(-\int_0^1 x^{n-1}\ln(1-x^2)\mathrm{d}x\right)\\ &= -\int_0^1\frac{\ln(1-x^2)}{x}\left(\sum_{n=1}^{\infty}(-x)^n\right)\mathrm{d}x\\ &= -\int_0^1\frac{\ln(1-x^2)}{x}\left(\frac{-x}{1+x}\right)\mathrm{d}x\\ &= \int_0^1\frac{\ln(1+x)}{1+x}\mathrm{d}x + \int_0^1\frac{\ln(1-x)}{1+x}\mathrm{d}x.\end{aligned}$$

The first integral is $\frac{1}{2}\ln^2(2)$ and the second is given in (3.17).
For a different approach, set $x = -1$ in the generating function (2.35).

4.7.2 $\sum_{n=1}^{\infty}\frac{H_{\frac{n}{2}}}{n^2}$

Show that

$$\sum_{n=1}^{\infty}\frac{H_{\frac{n}{2}}}{n^2} = \frac{11}{8}\zeta(3). \tag{4.149}$$

Solution. Employing the integral (4.148), we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_{\frac{n}{2}}}{n^2} &= \sum_{n=1}^{\infty}\frac{1}{n}\left(-\int_0^1 x^{n-1}\ln(1-x^2)\mathrm{d}x\right)\\ &= -\int_0^1\frac{\ln(1-x^2)}{x}\left(\sum_{n=1}^{\infty}\frac{x^n}{n}\right)\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \frac{\ln(1-x^2)\ln(1-x)}{x}\mathrm{d}x \\
&= \int_0^1 \frac{\ln^2(1-x)}{x}\mathrm{d}x + \int_0^1 \frac{\ln(1+x)\ln(1-x)}{x}\mathrm{d}x.
\end{aligned}
$$

These two integrals are given in (3.8) and (3.98).
For a different approach, see (4.18).

4.7.3 $\sum_{n=1}^{\infty} \frac{(-1)^n H_{\frac{n}{2}}}{n^2}$

Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_{\frac{n}{2}}}{n^2} = -\frac{3}{8}\zeta(3). \tag{4.150}$$

Solution. Applying the previous solution,

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^n H_{\frac{n}{2}}}{n^2} &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left(-\int_0^1 x^{n-1}\ln(1-x^2)\mathrm{d}x\right) \\
&= -\int_0^1 \frac{\ln(1-x^2)}{x}\left(\sum_{n=1}^{\infty}\frac{(-x)^n}{n}\right)\mathrm{d}x \\
&= \int_0^1 \frac{\ln(1-x^2)\ln(1+x)}{x}\mathrm{d}x \\
&= \int_0^1 \frac{\ln^2(1+x)}{x}\mathrm{d}x + \int_0^1 \frac{\ln(1+x)\ln(1-x)}{x}\mathrm{d}x.
\end{aligned}
$$

These two integrals are given in (3.20) and (3.98).
Check (4.19) for a different method.

4.7.4 $\sum_{n=1}^{\infty} \frac{H_{\frac{n}{2}}}{n^3}$

Show that

$$\sum_{n=1}^{\infty} \frac{H_{\frac{n}{2}}}{n^3} = -2\operatorname{Li}_4\left(\frac{1}{2}\right) + \frac{11}{4}\zeta(4) - \frac{7}{4}\ln(2)\zeta(3) + \frac{1}{2}\ln^2(2)\zeta(2) - \frac{1}{12}\ln^4(2). \tag{4.151}$$

Solution.

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{H_n-H_{\frac{n}{2}}}{n^3} &= \sum_{n=1}^{\infty}\frac{1}{n^2}\left(\frac{H_n-H_{\frac{n}{2}}}{n}\right)\\
&\quad\{\text{recall the integral (4.147)}\}\\
&= \sum_{n=1}^{\infty}\frac{1}{n^2}\left(\int_0^1 x^{n-1}\ln(1+x)\mathrm{d}x\right)\\
&= \int_0^1\frac{\ln(1+x)}{x}\left(\sum_{n=1}^{\infty}\frac{x^n}{n^2}\right)\mathrm{d}x\\
&= \int_0^1\frac{\ln(1+x)\operatorname{Li}_2(x)}{x}\mathrm{d}x\\
&= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^1 x^{n-1}\operatorname{Li}_2(x)\mathrm{d}x\\
&\quad\{\text{recall the result (3.66)}\}\\
&= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{\zeta(2)}{n}-\frac{H_n}{n^2}\right)\\
&= \zeta(2)\eta(2)+\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^3}. \qquad (4.152)
\end{aligned}$$

Since $\sum_{n=1}^{\infty}\frac{H_n}{n^3}=\frac{5}{4}\zeta(4)$ given in (4.4) and $\zeta(2)\eta(2)=\frac{5}{4}\zeta(4)$, we have

$$\sum_{n=1}^{\infty}\frac{H_{\frac{n}{2}}}{n^3}=-\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^3}.$$

This sum is calculated in (4.86).

4.7.5 $\sum_{n=1}^{\infty}\frac{(-1)^nH_{\frac{n}{2}}}{n^3}$

Show that

$$\sum_{n=1}^{\infty}\frac{(-1)^nH_{\frac{n}{2}}}{n^3}=2\operatorname{Li}_4\left(\frac{1}{2}\right)-\frac{27}{8}\zeta(4)+\frac{7}{4}\ln(2)\zeta(3)-\frac{1}{2}\ln^2(2)\zeta(2)+\frac{1}{12}\ln^4(2) \qquad (4.153)$$

Solution. Applying the previous solution,

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^n\frac{H_n-H_{\frac{n}{2}}}{n^3}&=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left(\frac{H_n-H_{\frac{n}{2}}}{n}\right)\\&=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left(\int_0^1 x^{n-1}\ln(1+x)\mathrm{d}x\right)\\&=\int_0^1\frac{\ln(1+x)}{x}\left(\sum_{n=1}^{\infty}\frac{(-x)^n}{n^2}\right)\mathrm{d}x\\&=-\int_0^1\frac{\ln(1+x)\operatorname{Li}_2(-x)}{x}\mathrm{d}x\\&=\frac{1}{2}\operatorname{Li}_2^2(-1)=\frac{5}{8}\zeta(4).\end{aligned}$$

The first sum on the LHS is given in (4.86).

4.7.6 $\sum_{n=1}^{\infty}\frac{H_nH_{\frac{n}{2}}}{n^2}$

Show that

$$\sum_{n=1}^{\infty}\frac{H_nH_{\frac{n}{2}}}{n^2}=-3\operatorname{Li}_4\left(\frac{1}{2}\right)+\frac{97}{16}\zeta(4)-\frac{21}{8}\ln(2)\zeta(3)+\frac{3}{4}\ln^2(2)\zeta(2)-\frac{1}{8}\ln^4(2).\tag{4.154}$$

Solution.

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n^2-H_nH_{\frac{n}{2}}}{n^2}&=\sum_{n=1}^{\infty}\frac{H_n}{n}\left(\frac{H_n-H_{\frac{n}{2}}}{n}\right)\\&=\sum_{n=1}^{\infty}\frac{H_n}{n}\left(\int_0^1 x^{n-1}\ln(1+x)\mathrm{d}x\right)\\&=\int_0^1\frac{\ln(1+x)}{x}\left(\sum_{n=1}^{\infty}\frac{H_n}{n}x^n\right)\mathrm{d}x\\&\quad\{\text{recall the generating function (2.6)}\}\\&=\int_0^1\frac{\ln(1+x)}{x}\left(\frac{1}{2}\ln^2(1-x)+\operatorname{Li}_2(x)\right)\mathrm{d}x\\&=\frac{1}{2}\int_0^1\frac{\ln(1+x)\ln^2(1-x)}{x}\mathrm{d}x+\int_0^1\frac{\ln(1+x)\operatorname{Li}_2(x)}{x}\mathrm{d}x\end{aligned}$$

$$\{\text{the second integral is simplified in (4.152)}\}$$

$$= \frac{1}{2}\int_0^1 \frac{\ln(1+x)\ln^2(1-x)}{x}\mathrm{d}x + \frac{5}{4}\zeta(4) + \sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^3}.$$

The solution finalizes on collecting the results (4.42), (4.86), and (3.116).

4.7.7 $\sum_{n=1}^{\infty}\frac{(-1)^n H_n H_{\frac{n}{2}}}{n^2}$

Show that

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n H_{\frac{n}{2}}}{n^2} = 5\operatorname{Li}_4\left(\frac{1}{2}\right) - \frac{21}{4}\zeta(4) + \frac{35}{8}\ln(2)\zeta(3) - \frac{5}{4}\ln^2(2)\zeta(2) + \frac{5}{24}\ln^4(2). \tag{4.155}$$

Solution.

$$\sum_{n=1}^{\infty}(-1)^n\frac{H_n^2 - H_n H_{\frac{n}{2}}}{n^2} = \sum_{n=1}^{\infty}(-1)^n\frac{H_n}{n}\left(\frac{H_n - H_{\frac{n}{2}}}{n}\right)$$

$$= \sum_{n=1}^{\infty}(-1)^n\frac{H_n}{n}\left(\int_0^1 x^{n-1}\ln(1+x)\mathrm{d}x\right)$$

$$= \int_0^1 \frac{\ln(1+x)}{x}\left(\sum_{n=1}^{\infty}\frac{H_n}{n}(-x)^n\right)\mathrm{d}x$$

$$\{\text{replace } x \text{ with } -x \text{ in (2.6)}\}$$

$$= \int_0^1 \frac{\ln(1+x)}{x}\left(\frac{1}{2}\ln^2(1+x) + \operatorname{Li}_2(-x)\right)\mathrm{d}x$$

$$= \frac{1}{2}\int_0^1 \frac{\ln^3(1+x)}{x}\mathrm{d}x + \int_0^1 \frac{\ln(1+x)\operatorname{Li}_2(-x)}{x}\mathrm{d}x$$

$$= \frac{1}{2}\int_0^1 \frac{\ln^3(1+x)}{x}\mathrm{d}x - \frac{1}{2}\operatorname{Li}_2^2(-x)\Big|_0^1$$

$$= \frac{1}{2}\int_0^1 \frac{\ln^3(1+x)}{x}\mathrm{d}x - \frac{5}{16}\zeta(4).$$

Collect the values (4.88) and (3.20) to finish the solution.

4.7.8 $\sum_{n=1}^{\infty} \frac{(-1)^n H_{\frac{n}{2}}}{n^4}$

Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_{\frac{n}{2}}}{n^4} = \frac{1}{8}\zeta(2)\zeta(3) - \frac{25}{32}\zeta(5). \tag{4.156}$$

Solution.

$$\begin{aligned}
\sum_{n=1}^{\infty} (-1)^n \frac{H_n - H_{\frac{n}{2}}}{n^4} &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \left(\frac{H_n - H_{\frac{n}{2}}}{n} \right) \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \left(\int_0^1 x^{n-1} \ln(1+x) \mathrm{d}x \right) \\
&= \int_0^1 \frac{\ln(1+x)}{x} \left(\sum_{n=1}^{\infty} \frac{(-x)^n}{n^3} \right) \mathrm{d}x \\
&= -\int_0^1 \frac{\ln(1+x)\operatorname{Li}_3(-x)}{x} \mathrm{d}x \\
&\overset{\text{IBP}}{=} -\frac{3}{8}\zeta(2)\zeta(3) + \int_0^1 \frac{\operatorname{Li}_2^2(-x)}{x} \mathrm{d}x.
\end{aligned}$$

Recall the relation involving the latter integral from (3.129),

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_{\frac{n}{2}}}{n^4} = -\frac{1}{4}\zeta(2)\zeta(3) - \frac{7}{8}\sum_{n=1}^{\infty} \frac{H_n}{n^4} - \sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n^4}.$$

Collecting the results (4.4) and (4.93) completes the solution.
For a different solution, see (4.19).

4.8 Harmonic Series with $\binom{2n}{n}$ in the Numerator

4.8.1 $\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{4^n} \frac{H_n}{n}$

Show that

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{4^n} \frac{H_n}{n} = 2\zeta(2). \tag{4.157}$$

Solution (i). By Taylor series, we have

$$\frac{1}{\sqrt{1-x}}=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{4^n}x^n=1+\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{4^n}x^n$$

or

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{4^n}x^n=\frac{1-\sqrt{1-x}}{\sqrt{1-x}}. \tag{4.158}$$

Using this identity, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{4^n}\frac{H_n}{n}&=\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{4^n}\left(-\int_0^1 x^{n-1}\ln(1-x)\mathrm{d}x\right)\\&=-\int_0^1\frac{\ln(1-x)}{x}\left(\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{4^n}x^n\right)\mathrm{d}x\\&=-\int_0^1\frac{\ln(1-x)}{x}\left(\frac{1-\sqrt{1-x}}{\sqrt{1-x}}\right)\mathrm{d}x\\&=-\underbrace{\int_0^1\frac{\ln(1-x)}{x\sqrt{1-x}}\mathrm{d}x}_{\sqrt{1-x}=y}+\underbrace{\int_0^1\frac{\ln(1-x)}{x}\mathrm{d}x}_{1-x=y}\\&=-4\int_0^1\frac{\ln(y)}{1-y^2}\mathrm{d}y+\int_0^1\frac{\ln(y)}{1-y}\mathrm{d}y\\&\left\{\text{write }\frac{1}{1-y^2}=\frac{1}{1-y}-\frac{y}{1-y^2}\text{ in the first integral}\right\}\\&=4\underbrace{\int_0^1\frac{y\ln(y)}{1-y^2}\mathrm{d}y}_{y\to\sqrt{y}}-3\int_0^1\frac{\ln(y)}{1-y}\mathrm{d}y\\&=-2\int_0^1\frac{\ln(y)}{1-y}\mathrm{d}y\\&=2\zeta(2).\end{aligned}$$

Solution (ii). Setting $x=1$ in (2.38):

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{4^n}\frac{H_n}{n}x^n=2\operatorname{Li}_2\left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right),$$

we have

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{4^n}\frac{H_n}{n}=2\operatorname{Li}_2(1)=2\zeta(2).$$

4.8.2 $\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{4^n} \frac{(-1)^n H_n}{n}$

Show that

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{4^n} \frac{(-1)^n H_n}{n} = 2\operatorname{Li}_2\left(2\sqrt{2}-3\right). \tag{4.159}$$

Solution. Set $x = -1$ in (2.38),

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{4^n} \frac{(-1)^n H_n}{n} = 2\operatorname{Li}_2\left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right) = 2\operatorname{Li}_2\left(2\sqrt{2}-3\right).$$

4.8.3 $\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{4^n} \frac{H_n}{n^2}$

Show that

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{4^n} \frac{H_n}{n^2} = \frac{9}{2}\zeta(3) - 4\ln(2)\zeta(2). \tag{4.160}$$

Solution (i).

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{4^n} \frac{x^n}{n} &= \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{4^n} \left(\int x^{n-1} \mathrm{d}x\right) \\
&= \int \frac{1}{x} \left(\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{4^n} x^n\right) \mathrm{d}x \\
&\quad \{\text{recall the generating function (4.158)}\} \\
&= \int \frac{1}{x} \left(\frac{1-\sqrt{1-x}}{\sqrt{1-x}}\right) \mathrm{d}x \\
&\overset{\sqrt{1-x}=y}{=} -2\int \frac{\mathrm{d}y}{1+y} \\
&= -2\ln(1+y) \\
&= -2\ln(1+\sqrt{1-x}) + c,
\end{aligned}$$

where $c = 2\ln(2)$ if we set $x = 0$. Then, we have

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{4^n} \frac{x^n}{n} = -2\ln(1+\sqrt{1-x}) + 2\ln(2). \tag{4.161}$$

Using this identity, we get

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{4^n}\frac{H_n}{n^2}&=\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{4^n n}\left(\frac{H_n}{n}\right)\\&=\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{4^n n}\left(-\int_0^1 x^{n-1}\ln(1-x)\mathrm{d}x\right)\\&=-\int_0^1\frac{\ln(1-x)}{x}\left(\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{4^n}\frac{x^n}{n}\right)\mathrm{d}x\\&=-\int_0^1\frac{\ln(1-x)}{x}\left(-2\ln(1+\sqrt{1-x})+2\ln(2)\right)\mathrm{d}x\\&=2\underbrace{\int_0^1\frac{\ln(1+\sqrt{1-x})\ln(1-x)}{x}\mathrm{d}x}_{\sqrt{1-x}=y}-2\ln(2)\underbrace{\int_0^1\frac{\ln(1-x)}{x}\mathrm{d}x}_{1-x=y}\\&=8\int_0^1\frac{y\ln(1+y)\ln(y)}{1-y^2}\mathrm{d}y-2\ln(2)\int_0^1\frac{\ln(y)}{1-y}\mathrm{d}y\\&=4\int_0^1\frac{\ln(1+y)\ln(y)}{1-y}\mathrm{d}y-4\int_0^1\frac{\ln(1+y)\ln(y)}{1+y}\mathrm{d}y-2\ln(2)\int_0^1\frac{\ln(y)}{1-y}\mathrm{d}y.\end{aligned}$$

These integrals are calculated in (3.103), (3.100), and (3.8) respectively.

Solution (ii). Set $x=1$ in (2.39),

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{4^n}\frac{H_n}{n^2}=-4\ln(2)\zeta(2)+2\zeta(3)-4\int_0^1\frac{\ln(1+t)\ln(1-t)}{t}\mathrm{d}t.$$

This integral is computed in (3.98).

4.8.4 $\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{4^n}\frac{H_n^{(2)}}{n}$

Show that

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{4^n}\frac{H_n^{(2)}}{n}=\frac{3}{2}\zeta(3).\tag{4.162}$$

Solution. Multiply both sides of (4.22):

$$\frac{H_n}{n^2}+\frac{H_n^{(2)}}{n}-\frac{\zeta(2)}{n}=\int_0^1 x^{n-1}\ln(x)\ln(1-x)\mathrm{d}x$$

by $\frac{\binom{2n}{n}}{4^n}$ then take the summation over $n \geq 1$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{4^n} \frac{H_n}{n^2} + \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{4^n} \frac{H_n^{(2)}}{n} - \zeta(2) \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{4^n} \frac{1}{n} \\
&= \int_0^1 \frac{\ln(x)\ln(1-x)}{x} \left(\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{4^n} x^n \right) \mathrm{d}x \\
&\qquad \{\text{recall the generating function (4.158)}\} \\
&= \int_0^1 \frac{\ln(x)\ln(1-x)}{x} \left(\frac{1-\sqrt{1-x}}{\sqrt{1-x}} \right) \mathrm{d}x \\
&= \underbrace{\int_0^1 \frac{\ln(x)\ln(1-x)}{x\sqrt{1-x}} \mathrm{d}x}_{1-x \to x} - \underbrace{\int_0^1 \frac{\ln(x)\ln(1-x)}{x} \mathrm{d}x}_{\text{IBP}} \\
&= \int_0^1 \frac{\ln(x)\ln(1-x)}{\sqrt{x}(1-x)} \mathrm{d}x - \frac{1}{2} \int_0^1 \frac{\ln^2(x)}{1-x} \mathrm{d}x. \\
&\qquad \{\text{substitute the results (3.102) and (3.8)}\} \\
&= 6\zeta(3) - 6\ln(2)\zeta(2).
\end{aligned}$$

The solution completes on collecting the result (4.160) and writing

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{4^n} \frac{1}{n} = 2\ln(2),$$

which follows from (4.161) on setting $x = 1$.

4.8.5 $\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{4^n} \frac{H_{2n}^{(2)}}{n}$

Show that

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{4^n} \frac{H_{2n}^{(2)}}{n} = 2\pi G + \frac{31}{8}\zeta(3). \tag{4.163}$$

Solution. By writing $\frac{1}{n} = 2\int_0^1 x^{2n-1}\mathrm{d}x$, we get

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{4^n} \frac{H_{2n}^{(2)} - \frac{1}{4}H_n^{(2)}}{n} &= \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{4^n} \left(H_{2n}^{(2)} - \frac{1}{4}H_n^{(2)} \right) \left(2\int_0^1 x^{2n-1}\mathrm{d}x \right) \\
&= \int_0^1 \frac{1}{x} \left(2\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{4^n} \left(H_{2n}^{(2)} - \frac{1}{4}H_n^{(2)} \right) x^{2n} \right) \mathrm{d}x
\end{aligned}$$

$$\{\text{recall the identity (2.58)}\}$$

$$= \int_0^1 \frac{1}{x}\left(\frac{\arcsin^2(x)}{\sqrt{1-x^2}}\right)\mathrm{d}x$$

$$\overset{x=\sin u}{=} \int_0^{\frac{\pi}{2}} u^2 \csc u \,\mathrm{d}u$$

$$\overset{\text{IBP}}{=} \underbrace{u^2 \ln\left(\tan\left(\frac{u}{2}\right)\right)\Bigg|_0^{\frac{\pi}{2}}}_{0} -2\int_0^{\frac{\pi}{2}} u \ln\left(\tan\left(\frac{u}{2}\right)\right)\mathrm{d}u$$

$$\{\text{use the Fourier series of } \ln(\tan x) \text{ given in (2.95)}\}$$

$$= -2\int_0^{\frac{\pi}{2}} u\left(-2\sum_{n=0}^{\infty}\frac{\cos((2n+1)u)}{2n+1}\right)\mathrm{d}u$$

$$= 4\sum_{n=0}^{\infty}\frac{1}{2n+1}\int_0^{\frac{\pi}{2}} u\cos((2n+1)u)\mathrm{d}u$$

$$= 4\sum_{n=0}^{\infty}\frac{1}{2n+1}\left(\frac{\pi\cos(n\pi)}{2(2n+1)} - \frac{\sin(n\pi)}{(2n+1)^2} - \frac{1}{(2n+1)^2}\right)$$

$$\left\{\text{use } \sum_{n=1}^{\infty} a_n\cos(n\pi) = \sum_{n=1}^{\infty}(-1)^n a_n \text{ given in (1.7)}\right\}$$

$$= 2\pi\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} - 4\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}$$

$$\{\text{the first sum is the definition of the Catalan's constant (1.162)}\}$$

$$\{\text{and the second sum can be obtained from (1.65)}\}$$

$$= 2\pi G + \frac{7}{2}\zeta(3).$$

The second sum on the LHS is given in (4.162) and the solution is complete.

4.8.6 $\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{4^n}\frac{H_n^2}{n}$

Show that

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{4^n}\frac{H_n^2}{n} = \frac{21}{2}\zeta(3). \tag{4.164}$$

Solution (i). By using the identity (2.62), we get

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{4^n}\frac{H_n^2+H_n^{(2)}}{n}&=\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{4^n}\left(\int_0^1 x^{n-1}\ln^2(1-x)\mathrm{d}x\right)\\
&=\int_0^1\frac{\ln^2(1-x)}{x}\left(\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{4^n}x^n\right)\mathrm{d}x\\
&\qquad\{\text{recall the generating function (4.158)}\}\\
&=\int_0^1\frac{\ln^2(1-x)}{x}\left(\frac{1-\sqrt{1-x}}{\sqrt{1-x}}\right)\mathrm{d}x\\
&=\underbrace{\int_0^1\frac{\ln^2(1-x)}{x\sqrt{1-x}}\mathrm{d}x}_{\sqrt{1-x}=y}-\underbrace{\int_0^1\frac{\ln^2(1-x)}{x}\mathrm{d}x}_{1-x=y}\\
&=8\int_0^1\frac{\ln^2(y)}{1-y^2}\mathrm{d}y-\int_0^1\frac{\ln^2(y)}{1-y}\mathrm{d}y.
\end{aligned}$$

Gathering the results (3.14), (3.8), and (4.162) completes the solution.

Solution (ii).

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{4^n}\frac{H_n^2}{n}&=\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{4^n}H_n\left(\frac{H_n}{n}\right)\\
&=\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{4^n}H_n\left(-\int_0^1 x^{n-1}\ln(1-x)\mathrm{d}x\right)\\
&=-\int_0^1\frac{\ln(1-x)}{x}\left(\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{4^n}H_nx^n\right)\mathrm{d}x\\
&\qquad\{\text{recall the generating function (2.37)}\}\\
&=-\int_0^1\frac{\ln(1-x)}{x}\left(\frac{2}{\sqrt{1-x}}\ln\left(\frac{1+\sqrt{1-x}}{2\sqrt{1-x}}\right)\right)\mathrm{d}x\\
&\overset{\sqrt{1-x}=y}{=}-8\int_0^1\frac{\ln(y)}{1-y^2}\ln\left(\frac{1+y}{2y}\right)\mathrm{d}y\\
&\overset{y=\frac{1-x}{1+x}}{=}4\underbrace{\int_0^1\frac{\ln^2(1-x)}{x}\mathrm{d}x}_{1-x=y}-4\int_0^1\frac{\ln(1-x)\ln(1+x)}{x}\mathrm{d}x\\
&=4\int_0^1\frac{\ln^2(y)}{1-y}\mathrm{d}x-4\int_0^1\frac{\ln(1-x)\ln(1+x)}{x}\mathrm{d}x.
\end{aligned}$$

These two integrals are given in (3.8) and (3.98).

4.8.7 $\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{4^n} \frac{H_n^2}{n^2}$

Show that

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{4^n} \frac{H_n^2}{n^2} = 32\operatorname{Li}_4\left(\frac{1}{2}\right) - 14\zeta(4) + 7\ln(2)\zeta(3) - 8\ln^2(2)\zeta(2) + \frac{4}{3}\ln^4(2). \tag{4.165}$$

Solution.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{4^n} \frac{H_n^2}{n^2} &= \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{4^n} \frac{H_n}{n} \left(\frac{H_n}{n}\right) \\
&= \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{4^n} \frac{H_n}{n} \left(-\int_0^1 x^{n-1}\ln(1-x)\mathrm{d}x\right) \\
&= -\int_0^1 \frac{\ln(1-x)}{x} \left(\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{4^n} \frac{H_n}{n} x^n\right) \mathrm{d}x \\
&\qquad \{\text{recall the generating function } (2.38)\} \\
&= -\int_0^1 \frac{\ln(1-x)}{x} \left(2\operatorname{Li}_2\left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)\right) \mathrm{d}x \\
&\overset{\sqrt{1-x}=y}{=} -8\int_0^1 \frac{y\ln(y)\operatorname{Li}_2\left(\frac{1-y}{1+y}\right)}{1-y^2}\mathrm{d}y \\
&\overset{\frac{1-y}{1+y}=x}{=} -4\int_0^1 \frac{(1-x)\ln\left(\frac{1-x}{1+x}\right)\operatorname{Li}_2(x)}{x(1+x)}\mathrm{d}x \\
&= -4\int_0^1 \frac{\ln(1-x)\operatorname{Li}_2(x)}{x}\mathrm{d}x + 4\int_0^1 \frac{\ln(1+x)\operatorname{Li}_2(x)}{x}\mathrm{d}x \\
&\quad -8\int_0^1 \frac{\ln(1+x)\operatorname{Li}_2(x)}{1+x}\mathrm{d}x + 8\int_0^1 \frac{\ln(1-x)\operatorname{Li}_2(x)}{1+x}\mathrm{d}x.
\end{aligned}$$

First integral:

$$\int_0^1 \frac{\ln(1-x)\operatorname{Li}_2(x)}{x}\mathrm{d}x = -\frac{1}{2}\operatorname{Li}_2^2(x)\Big|_0^1 = -\frac{1}{2}\zeta^2(2) = -\frac{5}{4}\zeta(4).$$

Second integral: We showed in (4.152)

$$\int_0^1 \frac{\ln(1+x)\operatorname{Li}_2(x)}{x}\mathrm{d}x = \zeta(2)\eta(2) + \sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n^3}$$

$$\{\text{collect the result (4.86)}\}$$
$$= 2\operatorname{Li}_4\left(\frac{1}{2}\right) - \frac{3}{2}\zeta(4) + \frac{7}{4}\ln(2)\zeta(3) - \frac{1}{2}\ln^2(2)\zeta(2) + \frac{1}{12}\ln^4(2).$$

Third integral: Perform integration by parts,

$$\int_0^1 \frac{\ln(1+x)\operatorname{Li}_2(x)}{1+x}\mathrm{d}x = \frac{1}{2}\ln^2(2)\zeta(2) + \frac{1}{2}\int_0^1 \frac{\ln^2(1+x)\ln(1-x)}{x}\mathrm{d}x$$
$$\{\text{this integral is given in (3.115)}\}$$
$$= \frac{1}{2}\ln^2(2)\zeta(2) - \frac{3}{16}\zeta(4).$$

Fourth integral: Make use of the dilogarithm reflection formula (1.80),

$$\int_0^1 \frac{\ln(1-x)\operatorname{Li}_2(x)}{1+x}\mathrm{d}x$$
$$= \int_0^1 \frac{\ln(1-x)[\zeta(2) - \ln(x)\ln(1-x) - \operatorname{Li}_2(1-x)]}{1+x}\mathrm{d}x$$
$$= \zeta(2)\int_0^1 \frac{\ln(1-x)}{1+x}\mathrm{d}x - \int_0^1 \frac{\ln(x)\ln^2(1-x)}{1+x}\mathrm{d}x$$
$$- \int_0^1 \frac{\ln(1-x)\operatorname{Li}_2(1-x)}{1+x}\mathrm{d}x$$
$$\{\text{these three integrals are given in (3.17), (4.144), and (4.143)}\}$$
$$= -\frac{5}{4}\zeta(4) - 2\ln(2)\zeta(3) + \frac{1}{2}\ln^2(2)\zeta(2) + 2\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n2^n} + \sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^2 2^n}$$
$$\{\text{collect the results (4.111) and (4.109)}\}$$
$$= 3\operatorname{Li}_4\left(\frac{1}{2}\right) - \frac{29}{16}\zeta(4) - \frac{1}{4}\ln^2(2)\zeta(2) + \frac{1}{8}\ln^4(2).$$

The solution completes on grouping the four integrals.

4.9 Harmonic Series with $\binom{2n}{n}$ in the Denominator

4.9.1 $\sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}} \frac{H_n}{n^2}$

Show that

$$\sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}} \frac{H_n}{n^2} = 6\ln(2)\zeta(2) + \frac{7}{2}\zeta(3). \tag{4.166}$$

Solution (i). Let $a = b = n$ in the beta function in (1.51):

$$\int_0^1 \frac{x^{a-1} + x^{b-1}}{(1+x)^{a+b}} \mathrm{d}x = \mathrm{B}(a, b),$$

we have

$$\int_0^1 \frac{2x^{n-1}}{(1+x)^{2n}} \mathrm{d}x = \frac{\Gamma^2(n)}{\Gamma(2n)} = \frac{2}{n\binom{2n}{n}}$$

or

$$\frac{1}{n\binom{2n}{n}} = \int_0^1 \frac{x^{n-1}}{(1+x)^{2n}} \mathrm{d}x = \int_0^1 \frac{1}{x} \left(\frac{x}{(1+x)^2}\right)^n \mathrm{d}x. \tag{4.167}$$

Using this result, we get

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}} \frac{H_n}{n^2} &= \sum_{n=1}^{\infty} 4^n \frac{H_n}{n} \frac{1}{n\binom{2n}{n}} \\
&= \sum_{n=1}^{\infty} 4^n \frac{H_n}{n} \left(\int_0^1 \frac{1}{x} \left(\frac{x}{(1+x)^2}\right)^n \mathrm{d}x \right) \\
&= \int_0^1 \frac{1}{x} \left(\sum_{n=1}^{\infty} \frac{H_n}{n} \left(\frac{4x}{(1+x)^2}\right)^n \right) \mathrm{d}x \\
&\qquad \{\text{make use of (2.6) for the sum}\} \\
&= \int_0^1 \frac{1}{x} \left(\operatorname{Li}_2\left(\frac{4x}{(1+x)^2}\right) + \frac{1}{2}\ln^2\left(1 - \frac{4x}{(1+x)^2}\right) \right) \mathrm{d}x \\
&\overset{\text{IBP}}{=} \int_0^1 \frac{2+2x}{x(1-x)} \ln(x) \ln\left(\frac{1-x}{1+x}\right) \mathrm{d}x \\
&= \int_0^1 \left(\frac{2}{x} + \frac{4}{1-x}\right) \ln(x) \ln\left(\frac{1-x}{1+x}\right) \mathrm{d}x
\end{aligned}$$

$$= 2\int_0^1 \frac{\ln(x)\ln(1-x)}{x}\mathrm{d}x + 4\underbrace{\int_0^1 \frac{\ln(x)\ln(1-x)}{1-x}\mathrm{d}x}_{1-x\to x}$$

$$-2\int_0^1 \frac{\ln(x)\ln(1+x)}{x}\mathrm{d}x - 4\int_0^1 \frac{\ln(x)\ln(1+x)}{1-x}\mathrm{d}x$$

$$= 6\int_0^1 \frac{\ln(x)\ln(1-x)}{x}\mathrm{d}x - 2\int_0^1 \frac{\ln(x)\ln(1+x)}{x}\mathrm{d}x - 4\int_0^1 \frac{\ln(x)\ln(1+x)}{1-x}\mathrm{d}x.$$

These integrals are given in (3.9), (3.12), and (3.103) respectively.

Solution (ii). Set $x = 1$ in (2.39),

$$\sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}} \frac{H_n}{n^2} = -4\ln(2)\zeta(2) + 2\zeta(3) - 4\int_0^1 \frac{\ln(1-t)\ln(1+t)}{t}\mathrm{d}t.$$

This integral is given in (3.98).

Solution (iii).

$$\sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}} \frac{H_n}{n^2} = \sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n} n}\left(\frac{H_n}{n}\right)$$

$$= \sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n} n}\left(-\int_0^1 x^{n-1}\ln(1-x)\mathrm{d}x\right)$$

$$= -\int_0^1 \frac{\ln(1-x)}{x}\left(\sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}}\frac{x^n}{n}\right)\mathrm{d}x$$

$$\{\text{replace } z \text{ with } \sqrt{x} \text{ in } (2.48)\}$$

$$= -\int_0^1 \frac{\ln(1-x)}{x}\left(2\frac{\sqrt{x}\arcsin(\sqrt{x})}{\sqrt{1-x}}\right)\mathrm{d}x$$

$$\overset{\sqrt{x}=\sin u}{=} -8\int_0^{\frac{\pi}{2}} u\ln(\cos u)\mathrm{d}u.$$

This integral is evaluated in (3.91).

4.9.2 $\sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}} \frac{H_{2n}}{n^2}$

Show that

$$\sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}} \frac{H_{2n}}{n^2} = 3\ln(2)\zeta(2) + \frac{35}{4}\zeta(3). \tag{4.168}$$

Solution.

$$\sum_{n=1}^{\infty}\frac{4^n}{\binom{2n}{n}}\frac{H_{2n}}{n^2}=\sum_{n=1}^{\infty}\frac{4^n}{\binom{2n}{n}n}\left(2\frac{H_{2n}}{2n}\right)$$

$$=\sum_{n=1}^{\infty}\frac{4^n}{\binom{2n}{n}n}\left(-2\int_0^1 x^{2n-1}\ln(1-x)\mathrm{d}x\right)$$

$$=-2\int_0^1\ln(1-x)\left(\sum_{n=1}^{\infty}\frac{4^n}{\binom{2n}{n}}\frac{x^{2n-1}}{n}\right)\mathrm{d}x$$

{recall the identity (2.48)}

$$=-2\int_0^1\ln(1-x)\left(2\frac{\arcsin(x)}{\sqrt{1-x^2}}\right)\mathrm{d}x$$

$$\overset{x=\sin u}{=}-4\int_0^{\frac{\pi}{2}}u\ln(1-\sin u)\mathrm{d}u$$

$$=-4\int_0^{\frac{\pi}{2}}u\ln\left(2\sin^2\left(\frac{\pi}{4}-\frac{u}{2}\right)\right)\mathrm{d}u$$

$$\overset{\frac{\pi}{4}-\frac{u}{2}=t}{=}\int_0^{\frac{\pi}{4}}(16t-4\pi)\ln(2\sin^2 t)\mathrm{d}t$$

$$=\ln(2)\int_0^{\frac{\pi}{4}}(16t-4\pi)\mathrm{d}t+2\int_0^{\frac{\pi}{4}}(16t-4\pi)\ln(\sin t)\mathrm{d}t$$

{use the Fourier series of $\ln(\sin t)$ given in (2.90)}

$$=\ln(2)\int_0^{\frac{\pi}{4}}(16t-4\pi)\mathrm{d}t+2\int_0^{\frac{\pi}{4}}(16t-4\pi)\left(-\ln(2)-\sum_{n=1}^{\infty}\frac{\cos(2nt)}{n}\right)\mathrm{d}t$$

$$=-\ln(2)\int_0^{\frac{\pi}{4}}(16t-4\pi)\mathrm{d}t-2\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\frac{\pi}{4}}(16t-4\pi)\cos(2nt)\mathrm{d}t$$

$$=-\ln(2)\left[-\frac{\pi^2}{2}\right]-2\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{4\cos(2nt)}{n^2}-\frac{2\pi\sin(2nt)}{n}+\frac{8t\sin(2nt)}{n}\right]_0^{\frac{\pi}{4}}$$

$$=3\ln(2)\zeta(2)-2\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{4\cos\left(\frac{n\pi}{2}\right)}{n^2}-\frac{4}{n^2}\right)$$

$$\left\{\text{make use of }\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi}{2}\right)=\sum_{n=1}^{\infty}(-1)^n a_{2n}\text{ given in (1.12)}\right\}$$

$$=3\ln(2)\zeta(2)-8\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)^3}+8\sum_{n=1}^{\infty}\frac{1}{n^3}$$

$$=3\ln(2)\zeta(2)+\frac{35}{4}\zeta(3).$$

4.9.3 $\sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}} \frac{H_n}{n^3}$

Show that

$$\sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}} \frac{H_n}{n^3} = -8\operatorname{Li}_4\left(\frac{1}{2}\right) + \zeta(4) + 8\ln^2(2)\zeta(2) - \frac{1}{3}\ln^4(2). \quad (4.169)$$

Solution.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}} \frac{H_n}{n^3} &= \sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n} n^2} \left(\frac{H_n}{n}\right) \\
&= \sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n} n^2} \left(-\int_0^1 x^{n-1}\ln(1-x)\mathrm{d}x\right) \\
&= -\int_0^1 \frac{\ln(1-x)}{x} \left(\sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}} \frac{x^n}{n^2}\right) \mathrm{d}x \\
&\quad \{\text{set } z = \sqrt{x} \text{ in (2.49) to get this}\} \\
&= -\int_0^1 \frac{\ln(1-x)}{x} \left(2\arcsin^2(\sqrt{x})\right) \mathrm{d}x \\
&\overset{\sqrt{x}=\sin\theta}{=} -8\int_0^{\frac{\pi}{2}} x^2 \cot x \ln(\cos x)\mathrm{d}x.
\end{aligned}$$

Let's recall the Fourier series of $\cot x \ln(\cos x)$ given in (2.107):

$$\cot x \ln(\cos x) = \sum_{n=1}^{\infty} (-1)^n \left(\int_0^1 \frac{1-t}{1+t} t^{n-1}\mathrm{d}t\right) \sin(2nx), \quad 0 < x < \pi.$$

On multiplying both sides by x^2 then integrating from $x = 0$ to $\frac{\pi}{2}$, we have

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} x^2 \cot x \ln(\cos x)\mathrm{d}x \\
&= \sum_{n=1}^{\infty} (-1)^n \left(\int_0^1 \frac{1-t}{1+t} t^{n-1}\mathrm{d}t\right) \left(\int_0^{\frac{\pi}{2}} x^2 \sin(2nx)\mathrm{d}x\right) \\
&= \sum_{n=1}^{\infty} (-1)^n \left(\int_0^1 \frac{1-t}{1+t} t^{n-1}\mathrm{d}t\right) \left(\frac{\cos(n\pi)}{4n^3} - \frac{3\zeta(2)\cos(n\pi)}{4n}\right. \\
&\qquad \left. -\frac{1}{4n^3} + \frac{\pi\sin(n\pi)}{4n^2}\right)
\end{aligned}$$

$$\left\{\text{remember that } \sum_{n=1}^{\infty} a_n(\cos n\pi) = \sum_{n=1}^{\infty} (-1)^n a_n\right\}$$

$$= \sum_{n=1}^{\infty} (-1)^n \left(\int_0^1 \frac{1-t}{1+t} t^{n-1} \mathrm{d}t\right)\left(\frac{(-1)^n}{4n^3} - \frac{3\zeta(2)(-1)^n}{4n} - \frac{1}{4n^3} + 0\right)$$

$$\{\text{change the order of integration and summation}\}$$

$$= \frac{1}{4}\int_0^1 \frac{1-t}{t(1+t)}\left(\sum_{n=1}^{\infty} \frac{t^n}{n^3} - \frac{3\zeta(2)t^n}{n} - \frac{(-t)^n}{n^3}\right)\mathrm{d}t$$

$$= \frac{1}{4}\int_0^1 \left(\frac{1}{t} - \frac{2}{1+t}\right)(\mathrm{Li}_3(t) + 3\zeta(2)\ln(1-t) - \mathrm{Li}_3(-t))\,\mathrm{d}t$$

$$= \frac{1}{4}\int_0^1 \frac{\mathrm{Li}_3(t) - \mathrm{Li}_3(-t)}{t}\mathrm{d}t - \frac{1}{2}\int_0^1 \frac{\mathrm{Li}_3(t) - \mathrm{Li}_3(-t)}{1+t}\mathrm{d}t$$

$$+\frac{3}{4}\zeta(2)\int_0^1 \frac{\ln(1-t)}{t}\mathrm{d}t - \frac{3}{2}\zeta(2)\int_0^1 \frac{\ln(1-t)}{1+t}\mathrm{d}t.$$

First integral:

$$\int_0^1 \frac{\mathrm{Li}_3(t) - \mathrm{Li}_3(-t)}{t}\mathrm{d}t = \mathrm{Li}_4(1) - \mathrm{Li}_4(-1) = \zeta(4) + \frac{7}{8}\zeta(4) = \frac{15}{8}\zeta(4).$$

Second integral: Apply integration by parts,

$$\int_0^1 \frac{\mathrm{Li}_3(t) - \mathrm{Li}_3(-t)}{1+t}\mathrm{d}t$$

$$= \frac{7}{4}\ln(2)\zeta(3) - \int_0^1 \frac{\ln(1+t)\,\mathrm{Li}_2(t)}{t}\mathrm{d}t + \int_0^1 \frac{\ln(1+t)\,\mathrm{Li}_2(-t)}{t}\mathrm{d}t$$

$$= \frac{7}{4}\ln(2)\zeta(3) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\int_0^1 t^{n-1}\,\mathrm{Li}_2(t)\mathrm{d}t - \frac{1}{2}\,\mathrm{Li}_2^2(-t)\Big|_0^1$$

$$\{\text{recall the result (3.66)}\}$$

$$= \frac{7}{4}\ln(2)\zeta(3) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left(\frac{\zeta(2)}{n} - \frac{H_n}{n^2}\right) - \frac{5}{16}\zeta(4)$$

$$= \frac{7}{4}\ln(2)\zeta(3) - \frac{5}{4}\zeta(4) - \sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n^3} - \frac{5}{16}\zeta(4)$$

$$\{\text{recall the result (4.86)}\}$$

$$= -2\,\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{25}{16}\zeta(4) + \frac{1}{2}\ln^2(2)\zeta(2) - \frac{1}{12}\ln^4(2).$$

Third integral:

$$\int_0^1 \frac{\ln(1-t)}{t}\mathrm{d}t = -\operatorname{Li}_2(1) = -\zeta(2).$$

Fourth integral: Let $1 - x = y$,

$$\begin{aligned}\int_0^1 \frac{\ln(1-t)}{1+t}\mathrm{d}t &= \int_0^1 \frac{\ln(y)}{2-y}\mathrm{d}y = \sum_{n=1}^{\infty} \frac{1}{2^n}\int_0^1 y^{n-1}\ln(y)\mathrm{d}y \\ &= -\sum_{n=1}^{\infty} \frac{1}{n^2 2^n} = -\operatorname{Li}_2\left(\frac{1}{2}\right) = \frac{1}{2}\ln^2(2) - \frac{1}{2}\zeta(2).\end{aligned}$$

Combining all four integrals reveals

$$\int_0^{\frac{\pi}{2}} x^2 \cot x \ln(\cos x)\mathrm{d}x = \operatorname{Li}_4\left(\frac{1}{2}\right) - \frac{1}{8}\zeta(4) - \ln^2(2)\zeta(2) + \frac{1}{24}\ln^4(2).$$

Plug this integral back to finish the solution.

4.9.4 $\sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}} \frac{H_n^{(2)}}{n^2}$

Show that

$$\sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}} \frac{H_n^{(2)}}{n^2} = 8\operatorname{Li}_4\left(\frac{1}{2}\right) - \zeta(4) + 4\ln^2(2)\zeta(2) + \frac{1}{3}\ln^4(2). \quad (4.170)$$

Solution (i). Set $x = 1$ in (2.54):

$$\begin{aligned}\frac{2}{3}\arcsin^4 x &= \sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}} \frac{H_{n-1}^{(2)} x^{2n}}{n^2} \\ &= \sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}} \frac{H_n^{(2)} x^{2n}}{n^2} - \sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}} \frac{x^{2n}}{n^4},\end{aligned}$$

we obtain

$$\sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}} \frac{H_n^{(2)}}{n^2} = \frac{15}{4}\zeta(4) + \sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}} \frac{1}{n^4}.$$

By writing $\frac{1}{n^2} = -4\int_0^1 t^{2n-1}\ln(t)\mathrm{d}t$, we get

$$\sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}} \frac{1}{n^4} = -4\sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}} \frac{1}{n^2}\int_0^1 t^{2n-1}\ln(t)\mathrm{d}t$$

$$= -4\int_0^1 \frac{\ln(t)}{t}\left(\sum_{n=1}^{\infty}\frac{4^n}{\binom{2n}{n}}\frac{t^{2n}}{n^2}\right)\mathrm{d}t$$

$$\{\text{recall the series expansion of } \arcsin^2 t \text{ in } (2.49)\}$$

$$= -8\int_0^1 \frac{\ln(t)\arcsin^2 t}{t}\mathrm{d}t$$

$$\overset{\text{IBP}}{=} 8\int_0^1 \frac{\ln^2(t)\arcsin t}{\sqrt{1-t^2}}\mathrm{d}t$$

$$\overset{t=\sin x}{=} 8\int_0^{\frac{\pi}{2}} x\ln^2(\sin x)\mathrm{d}x$$

$$\{\text{this integral is given in } (3.94)\}$$

$$= 8\operatorname{Li}_4\left(\frac{1}{2}\right) - \frac{19}{4}\zeta(4) + 4\ln^2(2)\zeta(2) + \frac{1}{3}\ln^4(2). \tag{4.171}$$

Solution (ii). Multiply both sides of (4.22):

$$\frac{H_n}{n^2} + \frac{H_n^{(2)}}{n} - \frac{\zeta(2)}{n} = \int_0^1 x^{n-1}\ln(x)\ln(1-x)\mathrm{d}x$$

by $\frac{4n}{n\binom{2n}{n}}$ then consider the summation,

$$\sum_{n=1}^{\infty}\frac{4^n}{\binom{2n}{n}}\frac{H_n}{n^3} + \sum_{n=1}^{\infty}\frac{4^n}{\binom{2n}{n}}\frac{H_n^{(2)}}{n^2} - \zeta(2)\sum_{n=1}^{\infty}\frac{4^n}{\binom{2n}{n}}\frac{1}{n^2}$$

$$= \int_0^1 \frac{\ln(x)\ln(1-x)}{x}\left(\sum_{n=1}^{\infty}\frac{4^n}{\binom{2n}{n}}\frac{x^n}{n}\right)\mathrm{d}x$$

$$\{\text{replace } z \text{ with } \sqrt{x} \text{ in } (2.48) \text{ to get the sum}\}$$

$$= \int_0^1 \frac{\ln(x)\ln(1-x)}{x}\left(\frac{2\sqrt{x}\arcsin\sqrt{x}}{\sqrt{1-x}}\right)\mathrm{d}x$$

$$\overset{\sqrt{x}=\sin\theta}{=} 16\int_0^{\frac{\pi}{2}} \theta\ln(\sin\theta)\ln(\cos\theta)\mathrm{d}\theta.$$

The first sum is given in (4.169). For the third, set $z = 1$ in (2.49),

$$\zeta(2)\sum_{n=1}^{\infty}\frac{4^n}{\binom{2n}{n}}\frac{1}{n^2} = \zeta(2)\left(\frac{\pi^2}{2}\right) = \frac{15}{2}\zeta(4).$$

For the integral, let $\theta \to \frac{\pi}{2} - \theta$ using $\cos(\frac{\pi}{2}-\theta) = \sin\theta$ and $\sin(\frac{\pi}{2}-\theta) = \cos\theta$,

$$\int_0^{\frac{\pi}{2}} \theta\ln(\sin\theta)\ln(\cos\theta)\mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2}-\theta\right)\ln(\cos\theta)\ln(\sin\theta)\mathrm{d}\theta$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \ln(\cos\theta)\ln(\sin\theta)\mathrm{d}\theta - \int_0^{\frac{\pi}{2}} \theta\ln(\cos\theta)\ln(\sin\theta)\mathrm{d}\theta.$$

$$\{\text{add the main integral to both sides then divide by } 2\}$$

$$= \frac{\pi}{4}\int_0^{\frac{\pi}{2}} \ln(\sin\theta)\ln(\cos\theta)\mathrm{d}\theta$$

$$\{\text{this integral is calculated in (3.93)}\}$$

$$= \frac{3}{4}\ln^2(2)\zeta(2) - \frac{15}{32}\zeta(4), \tag{4.172}$$

A different solution may be found in [32, p. 334]

4.9.5 $\sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}}\frac{H_n^2}{n^2}$

Show that

$$\sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}}\frac{H_n^2}{n^2} = -24\operatorname{Li}_4\left(\frac{1}{2}\right) + \frac{81}{2}\zeta(4) + 12\ln^2(2)\zeta(2) - \ln^4(2). \tag{4.173}$$

Solution. By using the identity (2.62), we get

$$\sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}}\frac{H_n^2 + H_n^{(2)}}{n^2} = \sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}n}\left(\frac{H_n^2 + H_n^{(2)}}{n}\right)$$

$$= \sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}n}\left(\int_0^1 x^{n-1}\ln^2(1-x)\mathrm{d}x\right)$$

$$= \int_0^1 \frac{\ln^2(1-x)}{x}\left(\sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}}\frac{x^n}{n}\right)\mathrm{d}x$$

$$\{\text{replace } z \text{ with } \sqrt{x} \text{ in (2.48) to get the sum}\}$$

$$= \int_0^1 \frac{\ln^2(1-x)}{x}\left(2\frac{\sqrt{x}\arcsin\sqrt{x}}{\sqrt{1-x}}\right)\mathrm{d}x$$

$$\overset{\sqrt{x}=\sin u}{=} 16\int_0^{\frac{\pi}{2}} u\ln^2(\cos u)\mathrm{d}u.$$

This integral and the second sum on the LHS are evaluated in (3.95) and (4.170).

4.9.6 $\sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}} \frac{H_{2n}}{n^3}$

Show that

$$\sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}} \frac{H_{2n}}{n^3} = -20\operatorname{Li}_4\left(\frac{1}{2}\right) + \frac{65}{8}\zeta(4) + 8\ln^2(2)\zeta(2) - \frac{5}{6}\ln^4(2). \quad (4.174)$$

Solution. Differentiate both sides of (4.167):

$$\frac{1}{n\binom{2n}{n}} = \int_0^1 \frac{1}{x}\left(\frac{x}{(1+x)^2}\right)^n \mathrm{d}x$$

with respect to n,

$$\begin{aligned}
\frac{d}{dn}\frac{1}{n\binom{2n}{n}} &= \frac{d}{dn}\int_0^1 \frac{1}{x}\left(\frac{x}{(1+x)^2}\right)^n \mathrm{d}x \\
&\{\text{use differentiation under the integral sign theorem (2.64)}\} \\
&= \int_0^1 \frac{1}{x}\frac{\partial}{\partial n}\left(\frac{x}{(1+x)^2}\right)^n \mathrm{d}x \\
&= \int_0^1 \frac{1}{x}\ln\left(\frac{x}{(1+x)^2}\right)\left(\frac{x}{(1+x)^2}\right)^n \mathrm{d}x.
\end{aligned}$$

Let's find the derivative of $\frac{1}{n\binom{2n}{n}}$. By the definition of the binomial coefficient:

$$\binom{a}{b} = \frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)},$$

we have

$$\frac{1}{n\binom{2n}{n}} = \frac{1}{n}\cdot\frac{\Gamma^2(n+1)}{\Gamma(2n+1)} = \frac{1}{n}\cdot\frac{(n\Gamma(n))^2}{2n\Gamma(2n)} = \frac{\Gamma^2(n)}{2\Gamma(2n)}.$$

Differentiate both sides,

$$\begin{aligned}
\frac{d}{dn}\frac{1}{n\binom{2n}{n}} &= \frac{d}{dn}\frac{\Gamma^2(n)}{2\Gamma(2n)} \\
&\{\text{use } \Gamma'(n) = \Gamma(n)\psi(n) \text{ given in (1.144)}\} \\
&= \frac{2\Gamma(2n)\Gamma^2(n)\psi(n) - 2\Gamma(2n)\Gamma^2(n)\psi(2n)}{2\Gamma^2(2n)}
\end{aligned}$$

$$= (\psi(n) - \psi(2n))\frac{\Gamma^2(n)}{\Gamma(2n)}$$

$$\{\text{use } \psi(n+1) = H_n - \gamma \text{ given in } (1.146)\}$$

$$= (H_{n-1} - \gamma - H_{2n-1} + \gamma)\frac{2}{n\binom{2n}{n}}$$

$$= \frac{2H_n}{n\binom{2n}{n}} - \frac{2H_{2n}}{n\binom{2n}{n}} - \frac{1}{n^2\binom{2n}{n}}.$$

Therefore, we have

$$\frac{2H_n}{n\binom{2n}{n}} - \frac{2H_{2n}}{n\binom{2n}{n}} - \frac{1}{n^2\binom{2n}{n}} = \int_0^1 \frac{1}{x}\ln\left(\frac{x}{(1+x)^2}\right)\left(\frac{x}{(1+x)^2}\right)^n \mathrm{d}x.$$

Now multiply both sides by $\frac{4^n}{2n^2}$ then consider the summation,

$$\sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}}\frac{H_n}{n^3} - \sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}}\frac{H_{2n}}{n^3} - \frac{1}{2}\sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}}\frac{1}{n^4}$$

$$= \frac{1}{2}\int_0^1 \frac{1}{x}\ln\left(\frac{x}{(1+x)^2}\right)\left[\sum_{n=1}^{\infty} \frac{\left(\frac{4x}{(1+x)^2}\right)^n}{n^2}\right]\mathrm{d}x$$

$$= \frac{1}{2}\int_0^1 \frac{1}{x}\ln\left(\frac{x}{(1+x)^2}\right)\left[\mathrm{Li}_2\left(\frac{4x}{(1+x)^2}\right)\right]\mathrm{d}x$$

$$\overset{\text{IBP}}{=} -\frac{5}{4}\zeta(4) - \frac{1}{2}\int_0^1 \left(\frac{1}{2}\ln^2(x) + 2\,\mathrm{Li}_2(-x)\right)\left[\frac{2(x-1)}{x(1+x)}\ln\left(\frac{1-x}{1+x}\right)\right]\mathrm{d}x$$

$$= -\frac{5}{4}\zeta(4) + \frac{1}{2}\int_0^1 \frac{\ln^2(x)\ln(1-x)}{x}\mathrm{d}x - \frac{1}{2}\int_0^1 \frac{\ln^2(x)\ln(1+x)}{x}\mathrm{d}x$$

$$+2\int_0^1 \frac{\ln(1-x)\,\mathrm{Li}_2(-x)}{x}\mathrm{d}x - 2\int_0^1 \frac{\ln(1+x)\,\mathrm{Li}_2(-x)}{x}\mathrm{d}x$$

$$-\int_0^1 \frac{\ln^2(x)\ln(1-x)}{1+x}\mathrm{d}x + \int_0^1 \frac{\ln^2(x)\ln(1+x)}{1+x}\mathrm{d}x$$

$$-4\int_0^1 \frac{\ln(1-x)\,\mathrm{Li}_2(-x)}{1+x}\mathrm{d}x + 4\int_0^1 \frac{\ln(1+x)\,\mathrm{Li}_2(-x)}{1+x}\mathrm{d}x.$$

The first and third sums on the LHS are computed in (4.169) and (4.171).

First integral: Expand $\ln(1-x)$ in series,

$$\int_0^1 \frac{\ln^2(x)\ln(1-x)}{x}\mathrm{d}x = -\sum_{n=1}^{\infty}\frac{1}{n}\int_0^1 x^{n-1}\ln^2(x)\mathrm{d}x$$

$$= -2\sum_{n=1}^{\infty}\frac{1}{n^4} = -2\zeta(4).$$

Second integral: Expand $\ln(1+x)$ in series,

$$\int_0^1 \frac{\ln^2(x)\ln(1+x)}{x}\mathrm{d}x = -\sum_{n=1}^{\infty}\frac{-1)^n}{n}\int_0^1 x^{n-1}\ln^2(x)\mathrm{d}x$$
$$= -2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4} = \frac{7}{4}\zeta(4).$$

Third integral: Expand $\operatorname{Li}_2(-x)$ in series,

$$\int_0^1 \frac{\ln(1-x)\operatorname{Li}_2(-x)}{x}\mathrm{d}x = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\int_0^1 x^{n-1}\ln(1-x)\mathrm{d}x$$
$$= -\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^3}$$
$$\{\text{this sum is given in (4.86)}\}$$
$$= -2\operatorname{Li}_4\left(\frac{1}{2}\right) + \frac{11}{4}\zeta(4) - \frac{7}{4}\ln(2)\zeta(3) + \frac{1}{2}\ln^2(2)\zeta(2) - \frac{1}{12}\ln^4(2).$$

Fourth integral:

$$\int_0^1 \frac{\ln(1+x)\operatorname{Li}_2(-x)}{x}\mathrm{d}x = -\frac{1}{2}\operatorname{Li}_2^2(-1) = -\frac{5}{16}\zeta(4).$$

The fifth and sixth integrals are given in (3.113) and (3.111) respectively.

Seventh integral: Expand $\frac{\operatorname{Li}_2(-x)}{1+x}$ in series given in (2.3),

$$\int_0^1 \frac{\ln(1-x)\operatorname{Li}_2(-x)}{1+x}\mathrm{d}x = -\sum_{n=1}^{\infty}(-1)^n H_{n-1}^{(2)}\int_0^1 x^{n-1}\ln(1-x)\mathrm{d}x$$
$$= -\sum_{n=1}^{\infty}(-1)^n\left(H_n^{(2)} - \frac{1}{n^2}\right)\left(-\frac{H_n}{n}\right)$$
$$= \sum_{n=1}^{\infty}\frac{(-1)^n H_n^{(2)} H_n}{n} - \sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^3}$$
$$\{\text{the first sum is given in (4.89)}\}$$
$$= \frac{15}{4}\zeta(4) - \frac{21}{8}\ln(2)\zeta(3) + \frac{3}{4}\ln^2(2)\zeta(2) - \frac{1}{6}\ln^4(2).$$

Eighth integral: Apply integration by parts,

$$\int_0^1 \frac{\ln(1+x)\operatorname{Li}_2(-x)}{1+x}\mathrm{d}x = -3\operatorname{Li}_4\left(\frac{1}{2}\right) - \frac{1}{4}\ln^2(2)\zeta(2) + \frac{1}{2}\int_0^1 \frac{\ln^3(1+x)}{x}\mathrm{d}x$$

$$\{\text{substitute the result (3.20)}\}$$

$$= 3\zeta(4) - \frac{21}{8}\ln(2)\zeta(3) + \frac{1}{2}\ln^2(2)\zeta(2) - \frac{1}{8}\ln^4(2).$$

Put all together to finalize the solution.

Remark: Usually the two *Mathematica* commands for approximating $\sum_{n=1}^{\infty} f(n)$:

```
NSum[f(n),{n,1,Infinity}]  and
NSum[f(n),{n,1,Infinity},WorkingPrecision->10]
```

don't give the right approximation for series involving the binomial coefficient due to the slow convergence. The following replacement works fine and with high accuracy:

```
major=Normal@Series[f(n),{n,Infinity,12}];
majorsum=Sum[major,{n,Infinity}];
majorsum+NSum[f(n)-major,{n,1,Infinity},NSumTerms->20,
WorkingPrecision->20,Method->"WynnEpsilon"]
```

On reaching the end of the book, I would like to say that there are still a wide range of results about the harmonic series left to be discovered by the reader by employing and manipulating the identities provided in the second chapter. Even though the book has presented different solutions for several problems, there are still more paths to take to reach the same results, since the realm of harmonic series is full of hidden secrets and magic.

Table of Mathematica Commands

$\Re\{\text{expression}\}$	`ComplexExpand[Re[expression]]`
$\Im\{\text{expression}\}$	`ComplexExpand[Im[expression]]`
$\sin(x)$	`Sin[x]`
$\sin(x)$	`Cos[x]`
$\tan(x)$	`Tan[x]`
$\sec(x)$	`Sec[x]`
$\csc(x)$	`Csc[x]`
$\cot(x)$	`Cot[x]`
$\arcsin(x)$	`ArcSin[x]`
$\arccos(x)$	`ArcCos[x]`
$\arctan(x)$	`ArcTan[x]`
$\binom{a}{b}$	`Binomial[a,b]`
$\Gamma(x)$	`Gamma[x]`
$\zeta(x)$	`Zeta[x]`
$\beta(x)$	`DirichletBeta[x]`
$\eta(x)$	`DirichletEta[x]`
$\ln(x)$	`Log[x]`
$\mathrm{Li}_a(x)$	`PolyLog[a,x]`
H_x	`HarmonicNumber[x]`
$H_x^{(a)}$	`HarmonicNumber[x,a]`
$\overline{H}_x$	`Log[2]-(-1)^x LerchPhi[-1,1,x+1]`
$\psi(x)$	`PolyGamma[0,x]`
$\psi^{(a)}(x)$	`PolyGamma[a,x]`
$\lim_{x\to a} f(x)$	`Limit[f(x),{x->a}]`
$\lim_{\substack{x\to a\\ y\to b}} f(x,y)$	`Limit[f(x,y),{x,y}->{a,b}]`
$\frac{d^a}{dx^a} f(x)$	`D[f(x),{x,a}]`
$\frac{\partial^5}{\partial x^2 \partial y^3} f(x,y)$	`D[f(x,y),{x,2},{y,3}]`
$\lim_{\substack{x\to a\\ y\to b}} \frac{\partial^5}{\partial x^2 \partial y^3} f(x,y)$	`Normal[Series[D[f(x,y),{x,2},{y,3}],{x,a,0},{y,b,0}]]//FullSimplify//Expand`

$\int_a^b f(x)\mathrm{d}x$	`Integrate[f(x),{x,a,b}]`
$\int_a^b f(x) \approx$	`NIntegrate[f(x),{x,a,b}]` (or) `NIntegrate[f(x),{x,a,b},WorkingPrecision->12]`
$\sum_{n=1}^{\infty} f(n)$	`Sum[f(n),{n,1,Infinity}]`

$\sum_{n=1}^{\infty} f(n) \approx$	`NSum[f(n),{n,1,Infinity}]` (or)
	`NSum[f(n),{n,1,Infinity},WorkingPrecision->12]` (or)
	`major=Normal@Series[f(n),{n,Infinity,12}];`
	`majorsum=Sum[major,{n,Infinity}];`
	`majorsum+NSum[f(n)-major,{n,1,Infinity},NSumTerms->20,WorkingPrecision->20,Method->"WynnEpsilon"]`

γ	`EulerGamma`
G	`Catalan`
e	`E`
π	`Pi`
$\sqrt{-1}$	`I`
B_n	`BernoulliB[n]`
E_n	`EulerE[n]`

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